Classical Stochastic and Quantum Conway's Game of Life

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2nd November 2023, ShanghaiAI Lectures

https://shanghai-lectures.github.io/programme/

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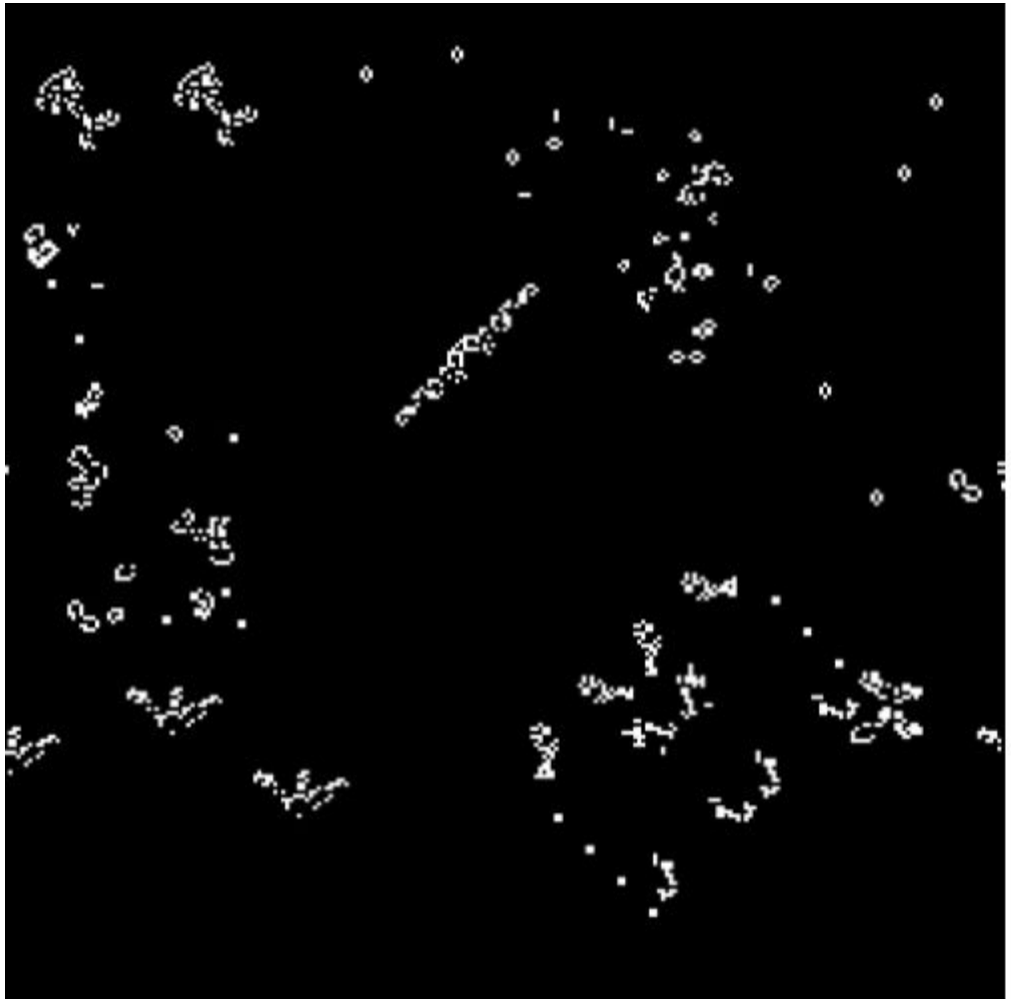
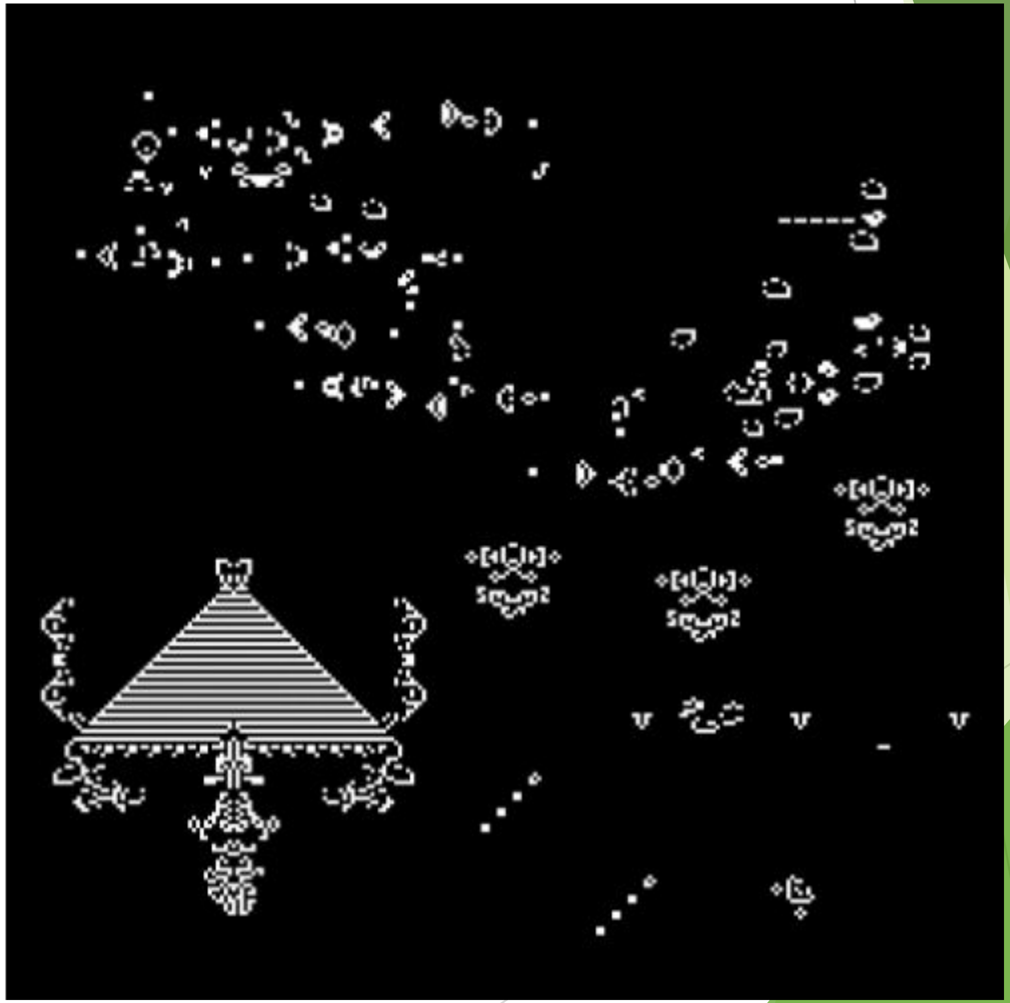
Lodz University of Technology
Institute of Physics

QHS
Quantum Hardware Systems

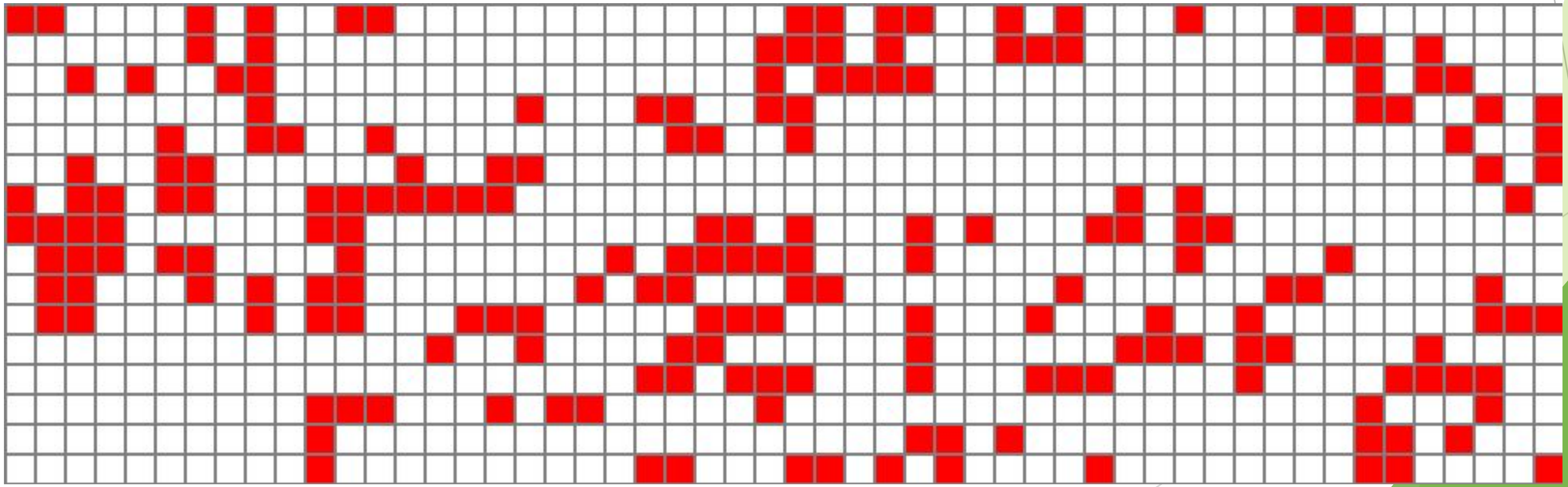
Introduction to Classic Conway's Game of Life





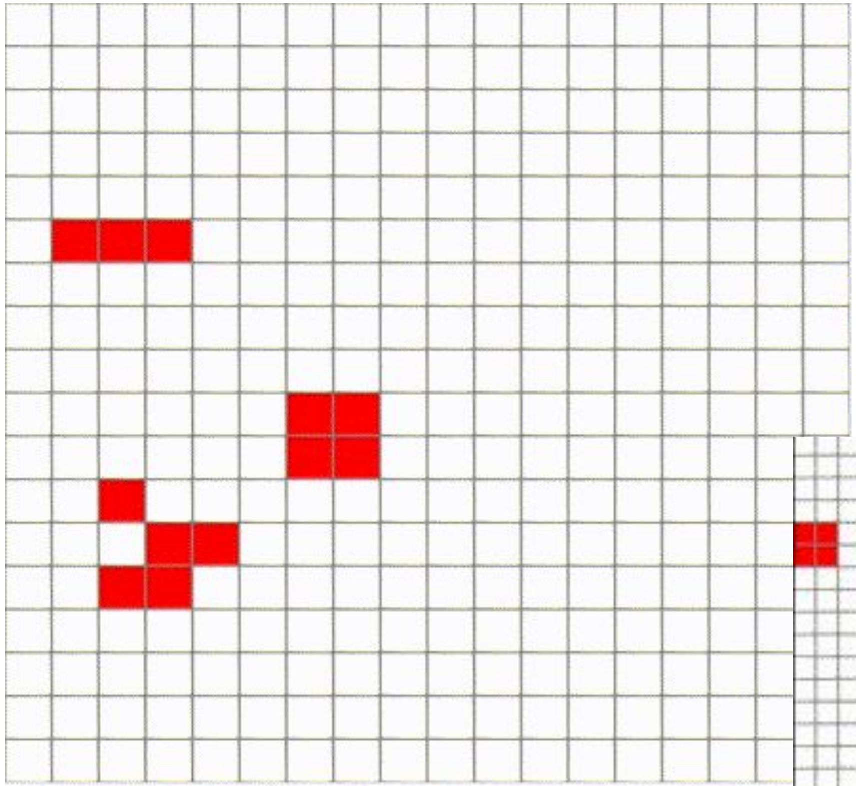
Rules of Classic Conway's Game of Life

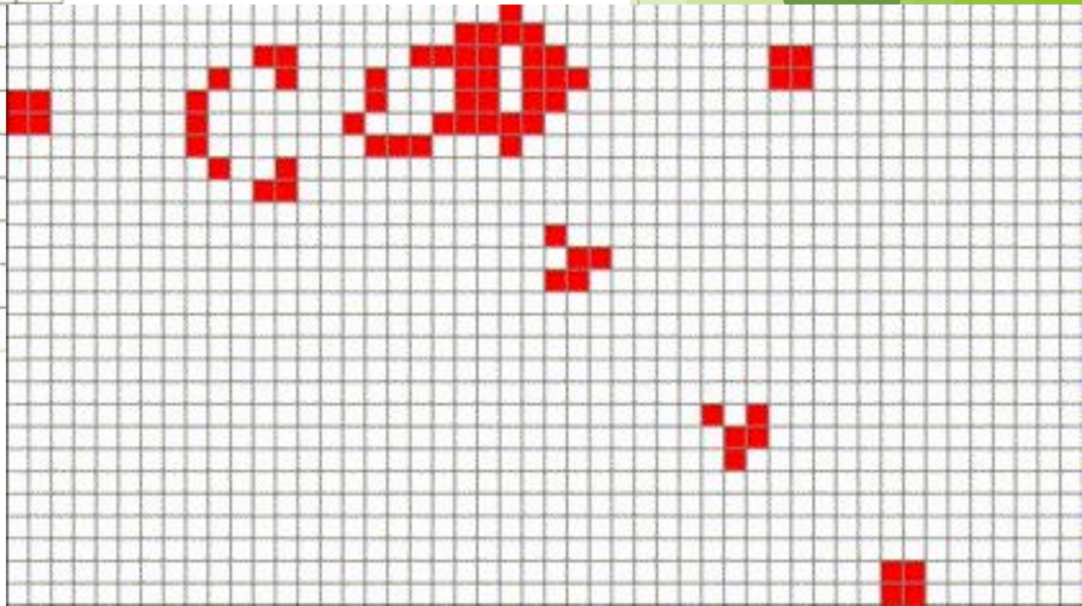
- Any live cell with two or three live neighbours survives.
- Any dead cell with three live neighbours becomes a live cell.
- All other live cells die in the next generation. Similarly, all other dead cells stay dead.



Dynamical stable structures in Classic Conway's Game of Life (CCGoL)

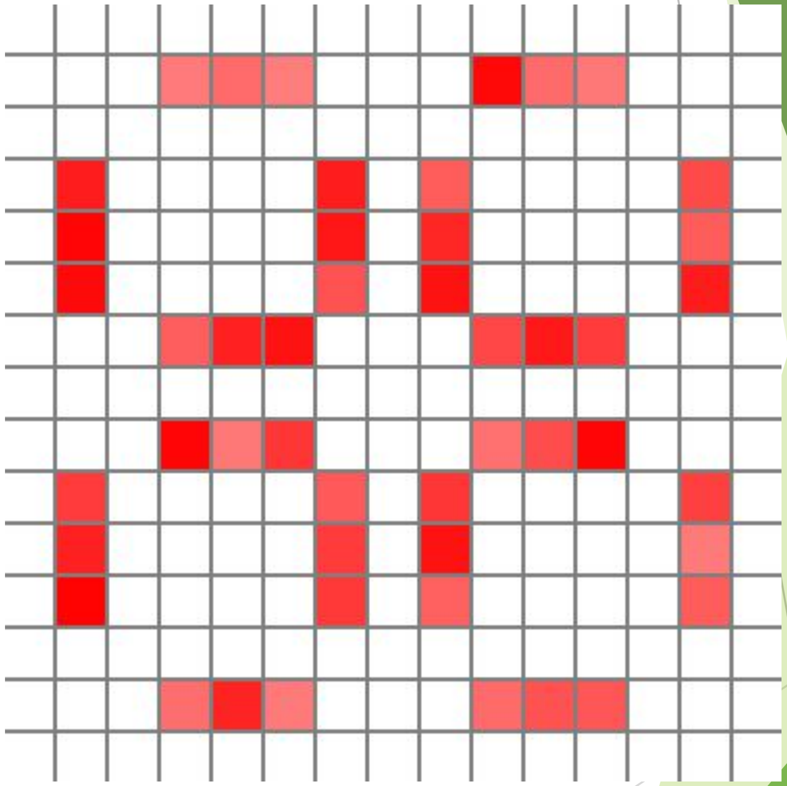
- Unstable structures
- Still lifes
- Oscillators
- Spaceships
- Guns



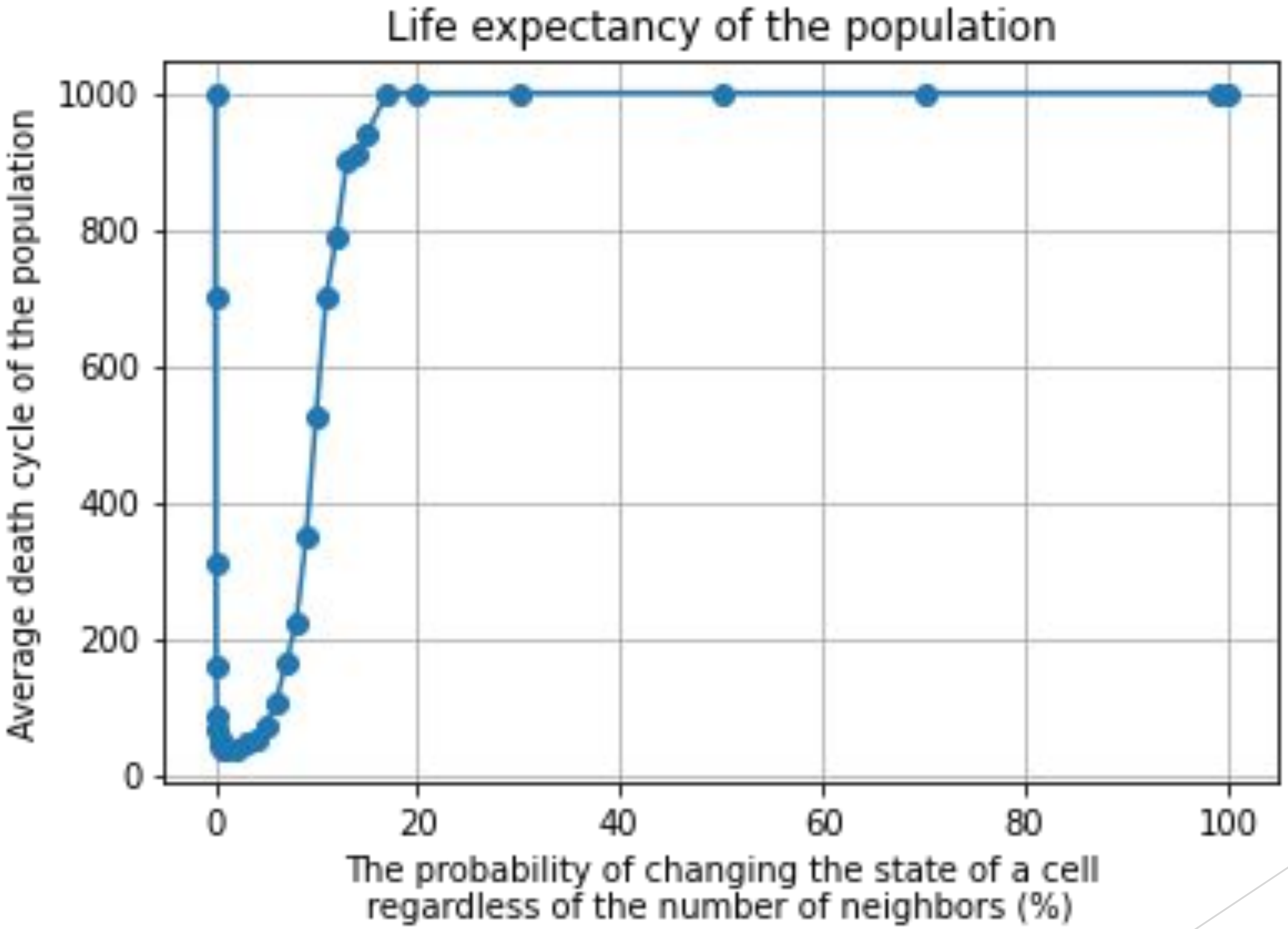


Rules in Stochastic Conway's Game of Life (SGoL)

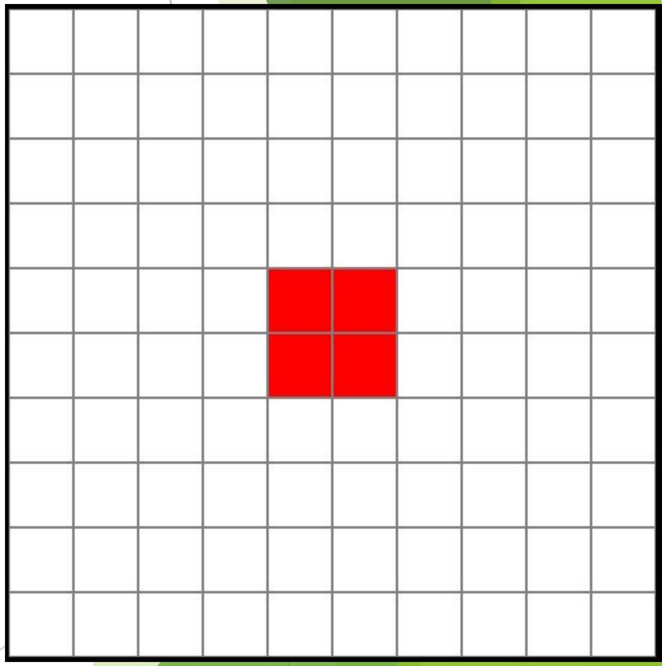
- Mass instead of two states
- Any dead cell with a sum of masses of its neighbours from the certain interval may become alive **with a given probability**.
- Any live cell with a sum of masses of its neighbours from the certain interval may stay alive **with a given probability**.



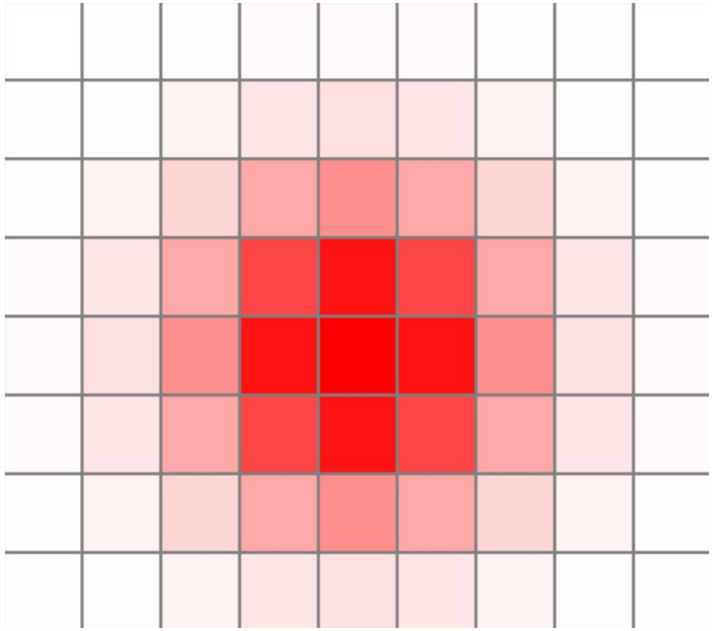
Mean life of cells on the board for different probabilities in SGoL



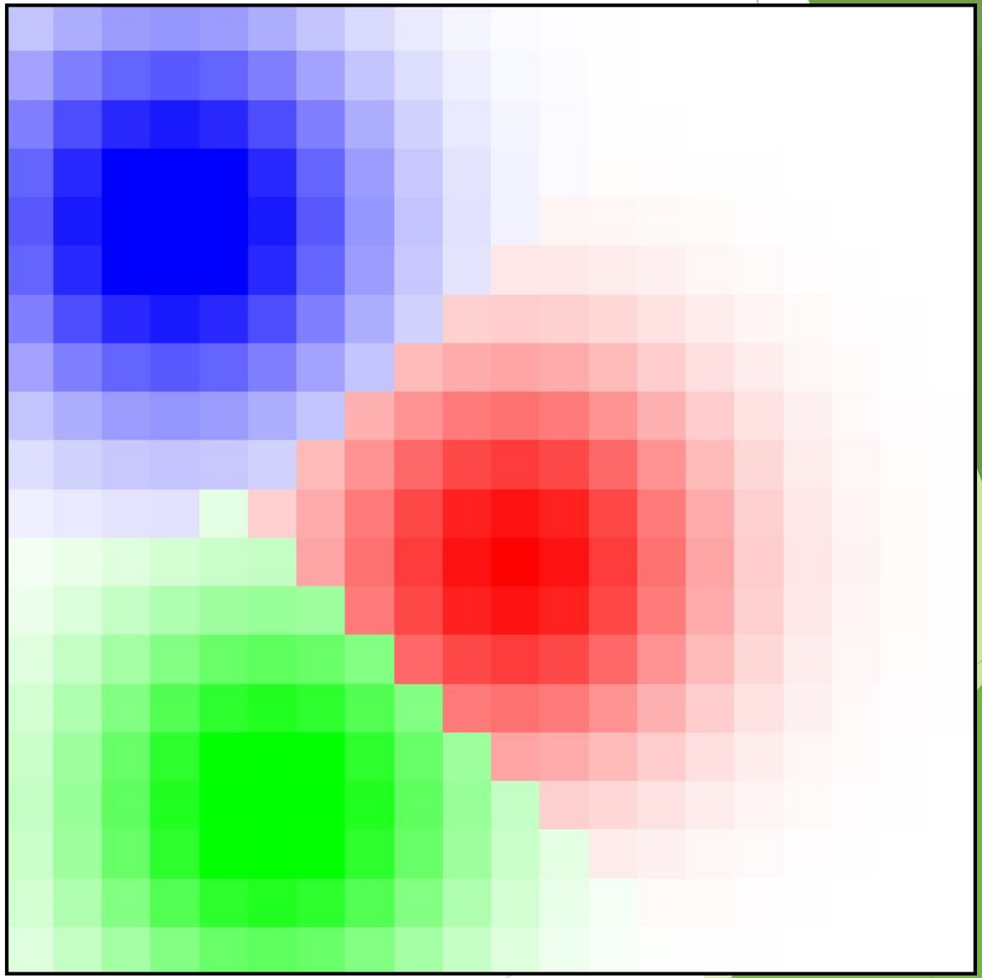
Initial structure



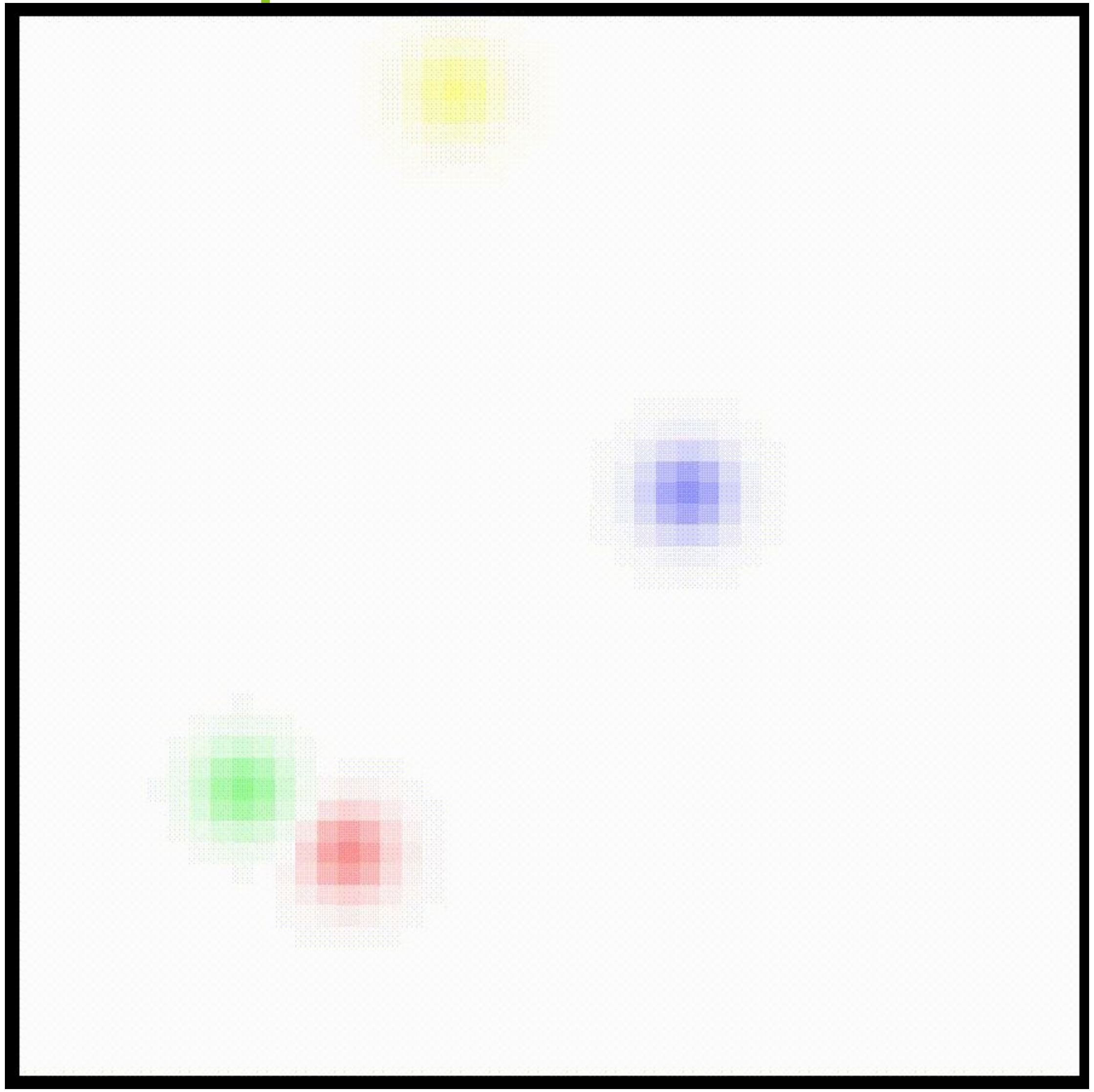
Generalization of SGoL to the case of N competing cellular automata species



$$f(x,y) = Ae^{-\left(\frac{(x-x_0)^2}{2\sigma_x^2}+\frac{(y-y_0)^2}{2\sigma_y^2}\right)}$$



Four species of cellular automata in SGoL (4 competing tribes)



Nash equilibrium exists?

Gauss range:

Mass of tribe:

Live cell dies by underpopulation for a number of neighbors less than: 0,3

Live cell dies by overpopulation for a number of neighbors greater than: 0,6

Dead cell comes alive for a number of neighbors of the same tribe greater than: 0,45 and less than: 0,7 and for a number of neighbors of the others tribes less than: 0,3

Temperature, energy and entropy

- Energy as mass in the first approximation
- Gibbs Entropy

$$dS = \frac{dQ}{T} \qquad T = \frac{dE}{dS}$$

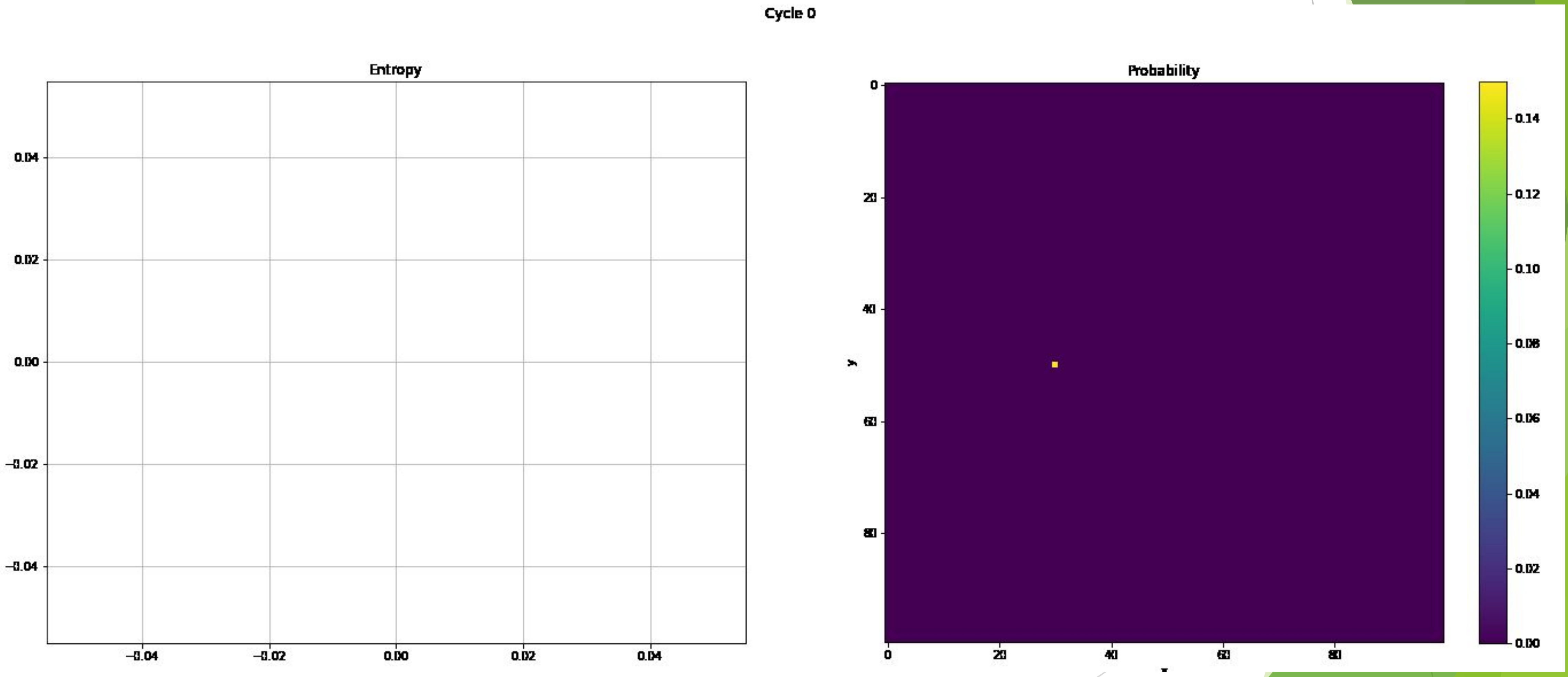
Q - heat exchange in thermodynamical cycle

T - derivative of energy with respect to entropy

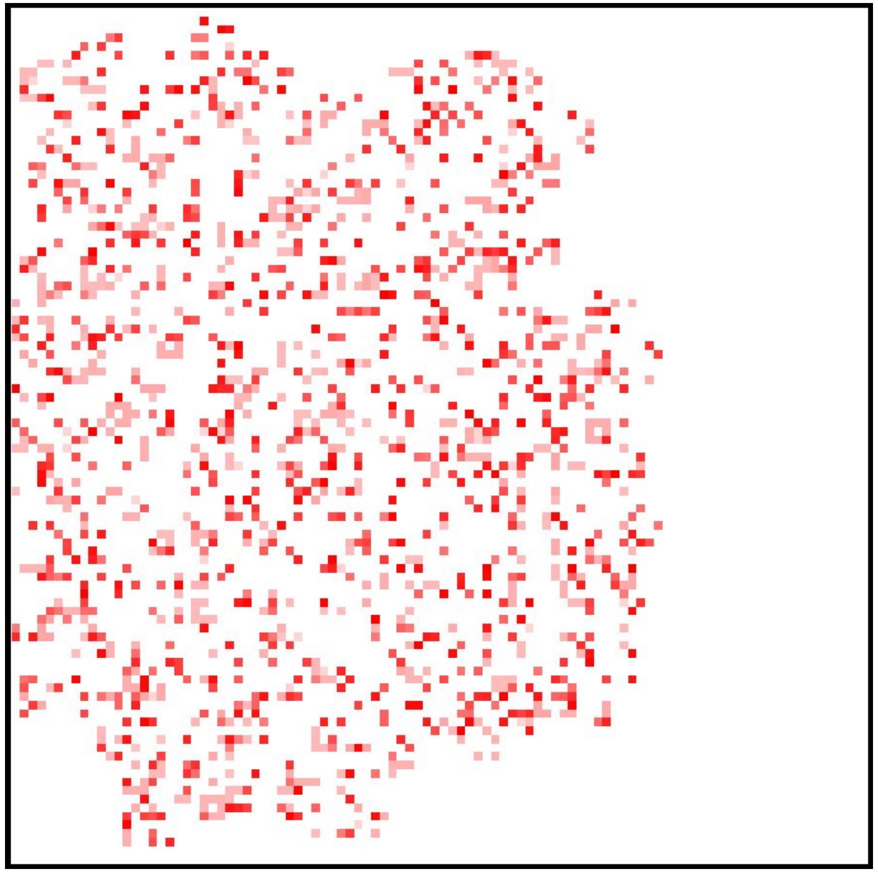
- Shannon Entropy

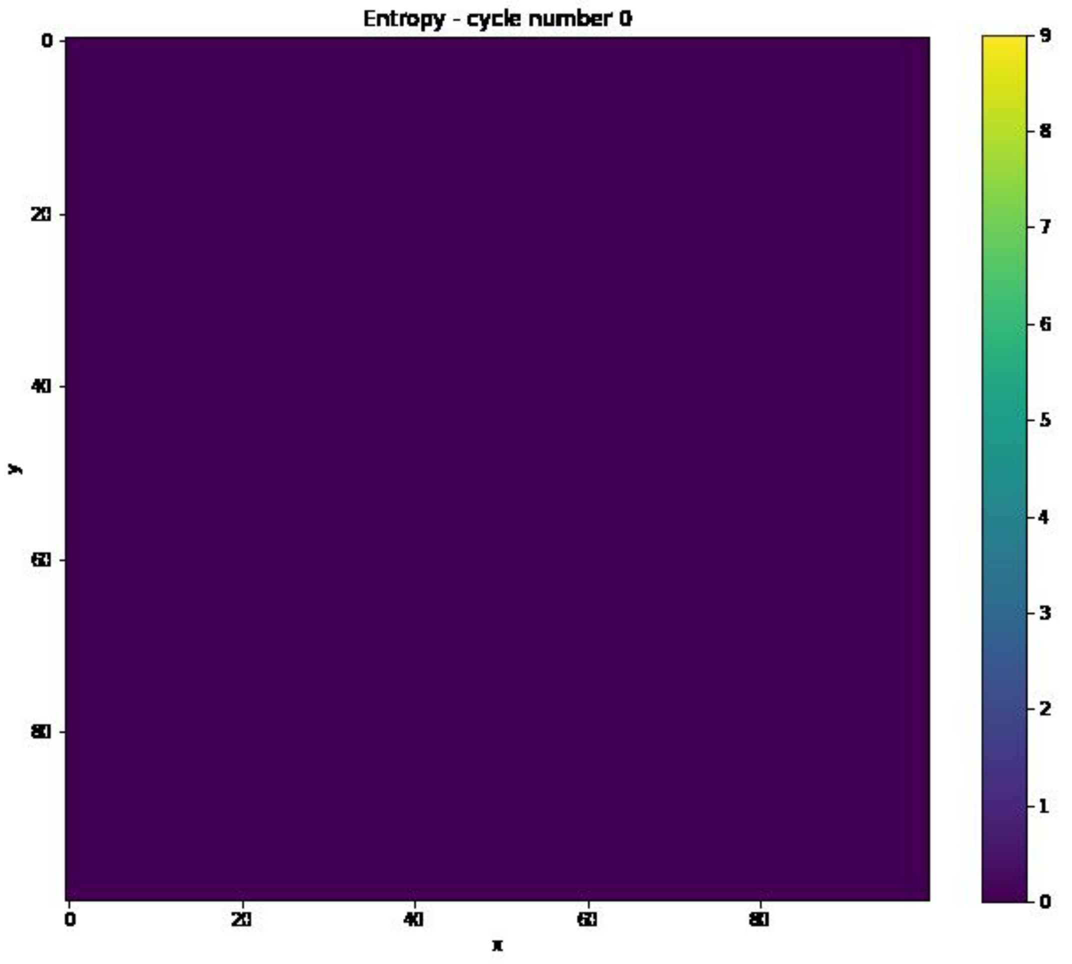
$$S = \mathbb{E}\left[-\log p(i,j)\right]$$

Evolution probability distribution with time in SGoL

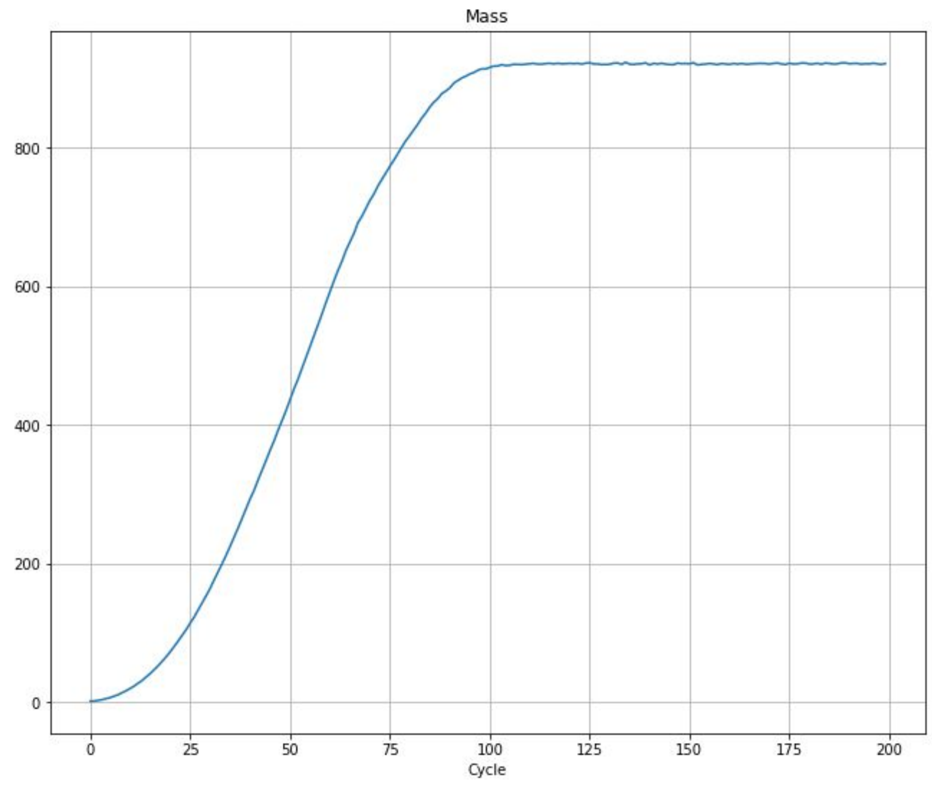


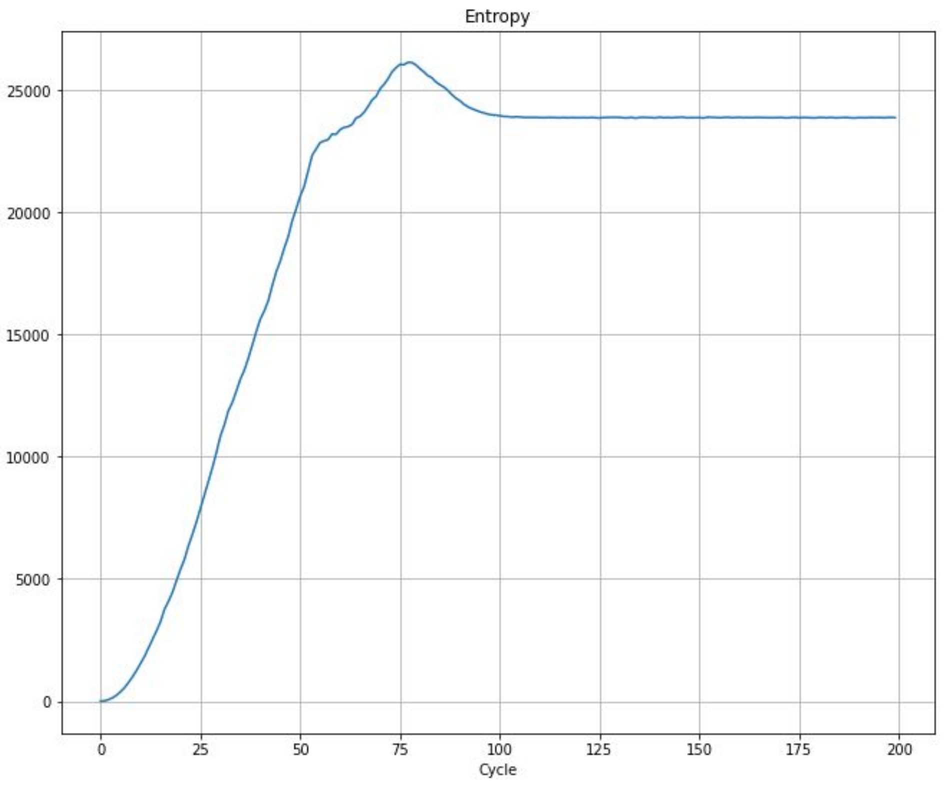
Evolution of entropy distribution with time in SGoL



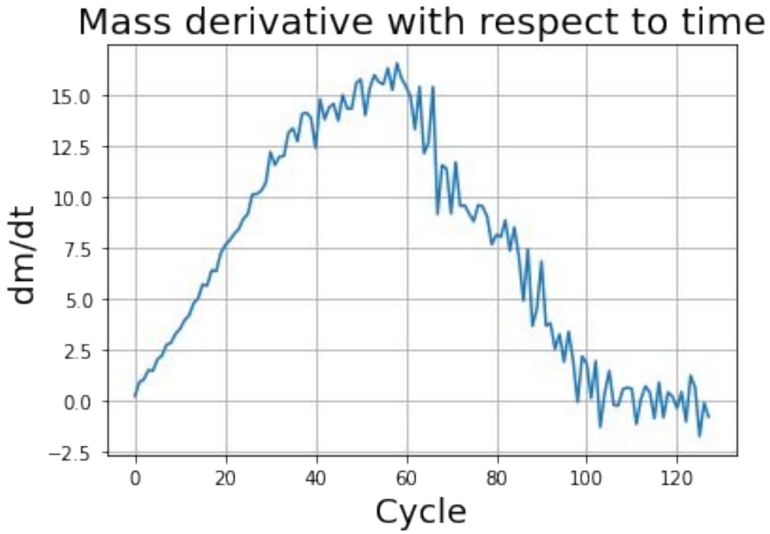


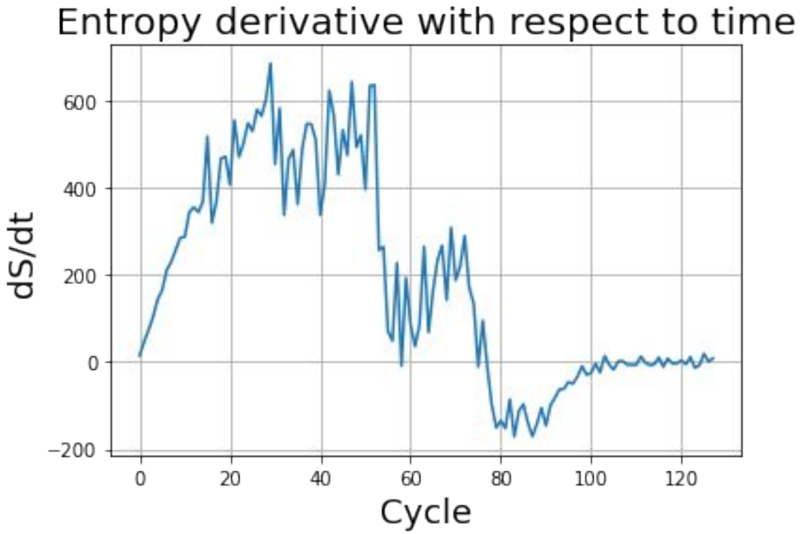
Mass (energy equivalence) and entropy evolution in time in SGoL

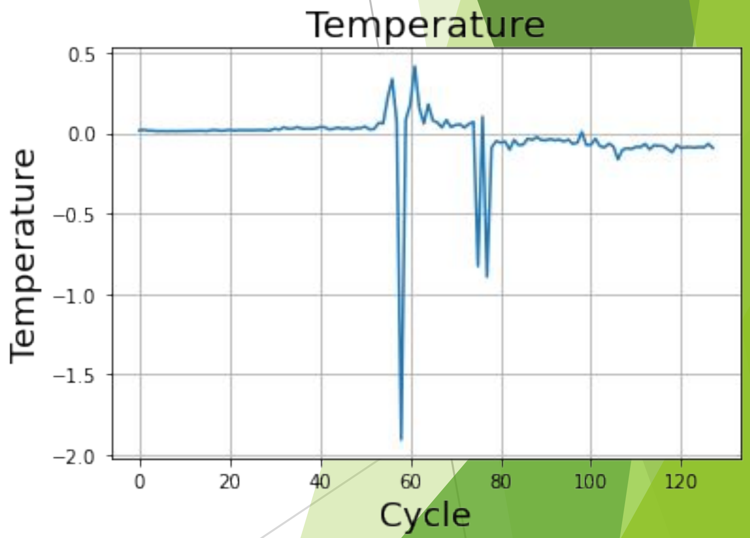




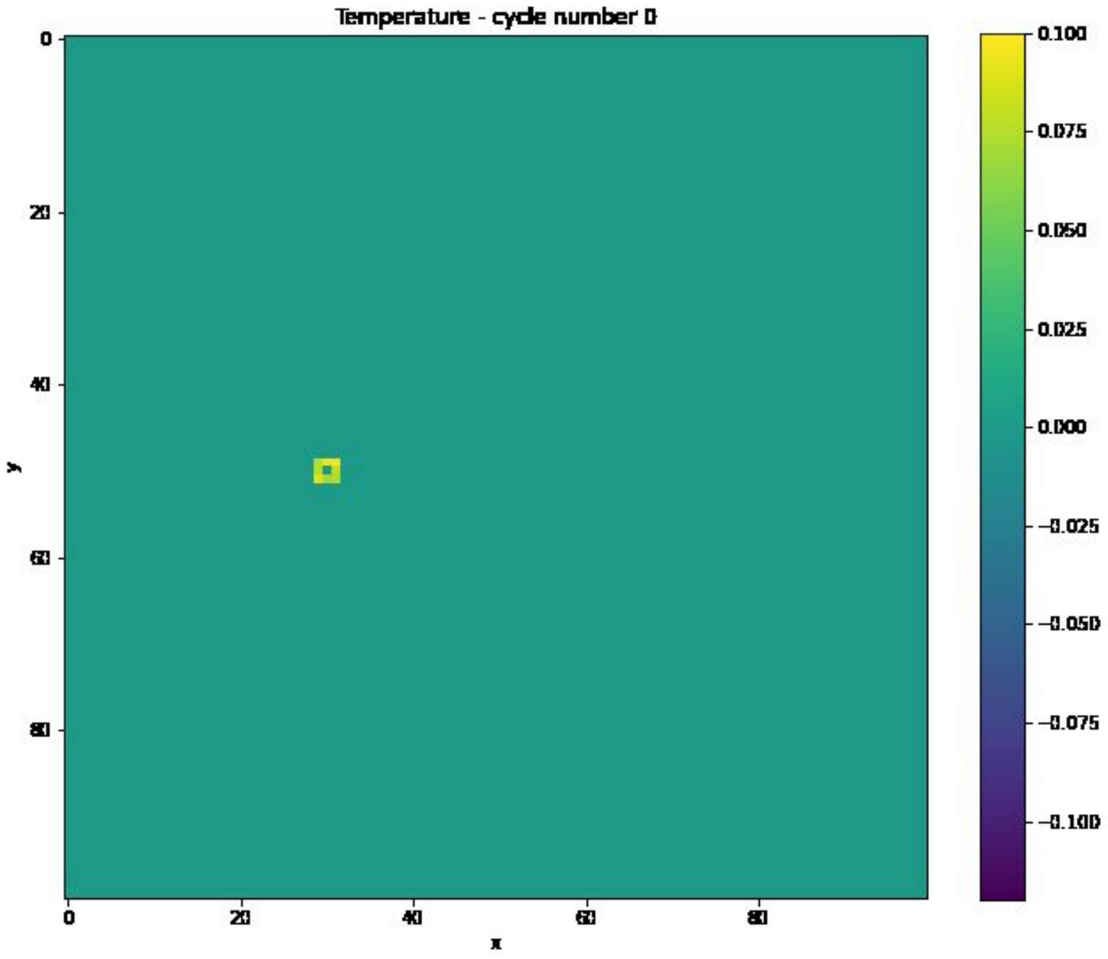
Negative temperature as a parameter describing SGoL for mass and entropy in equilibrium

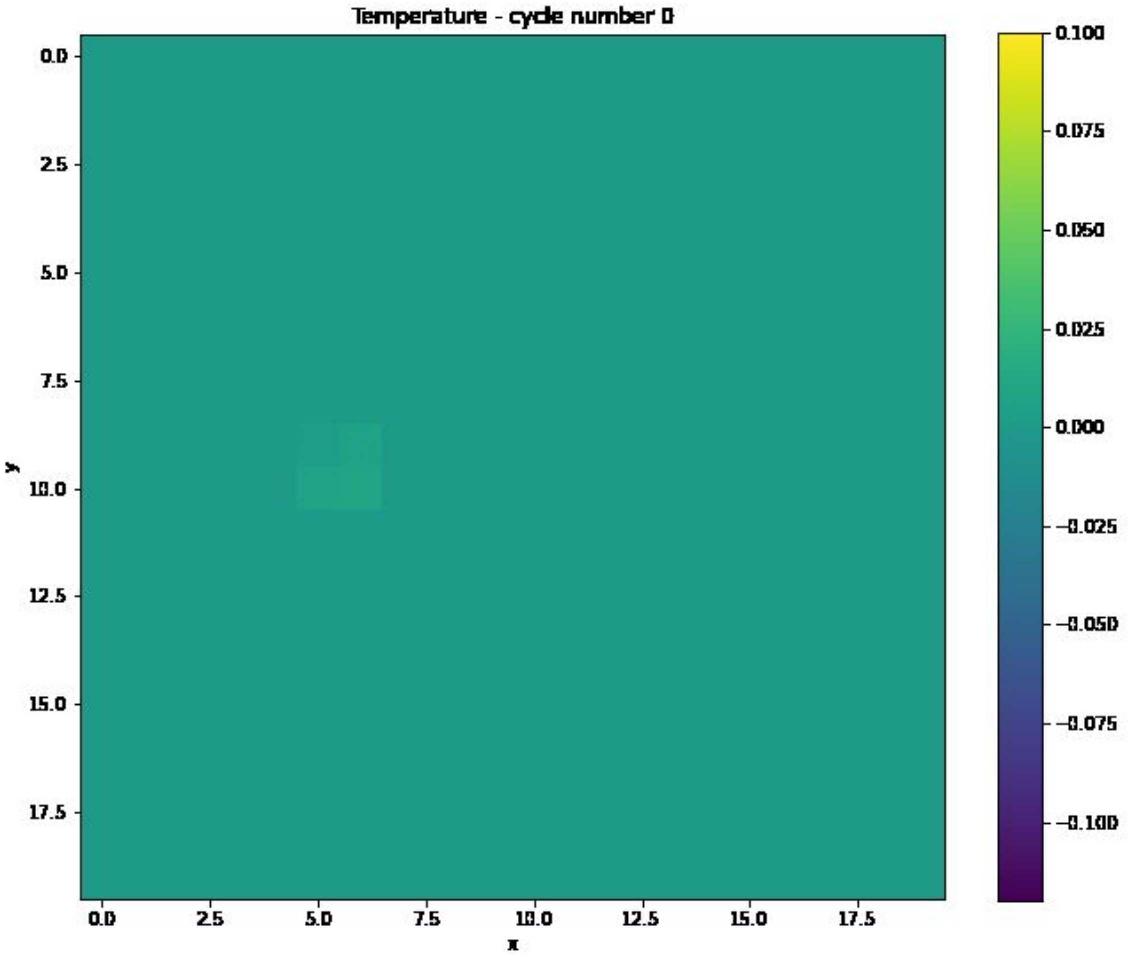




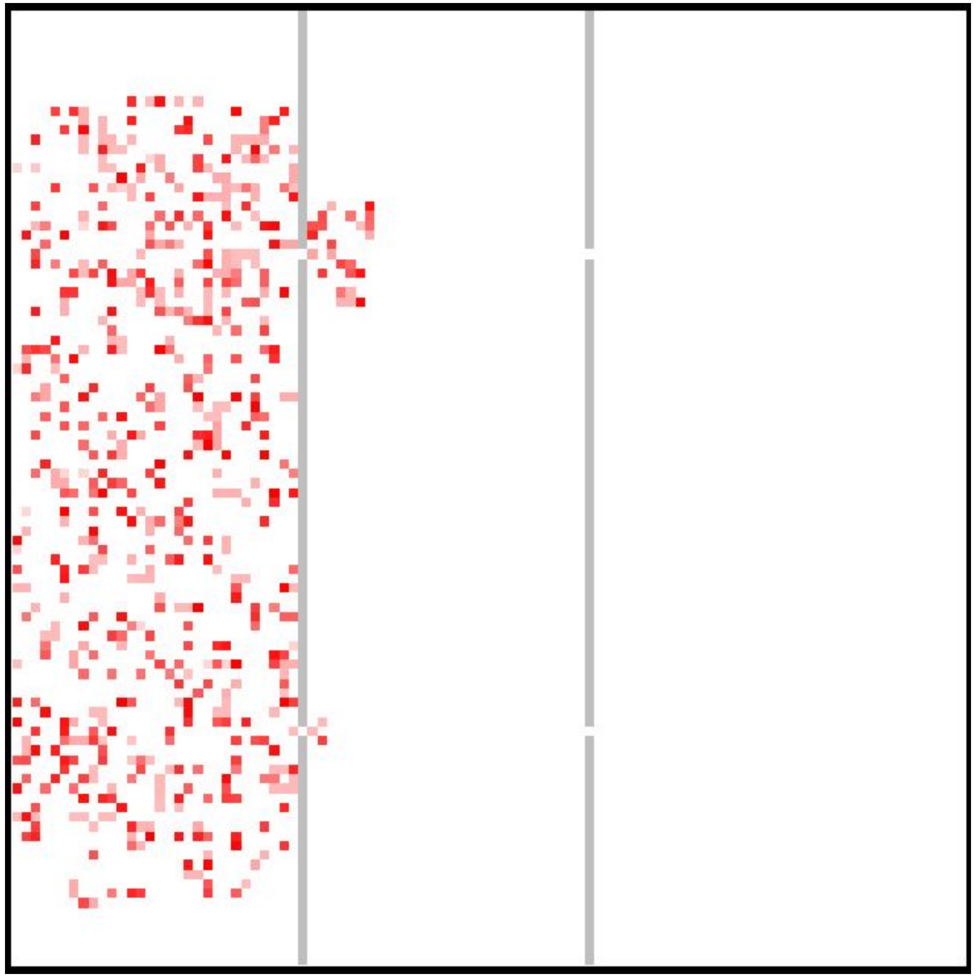


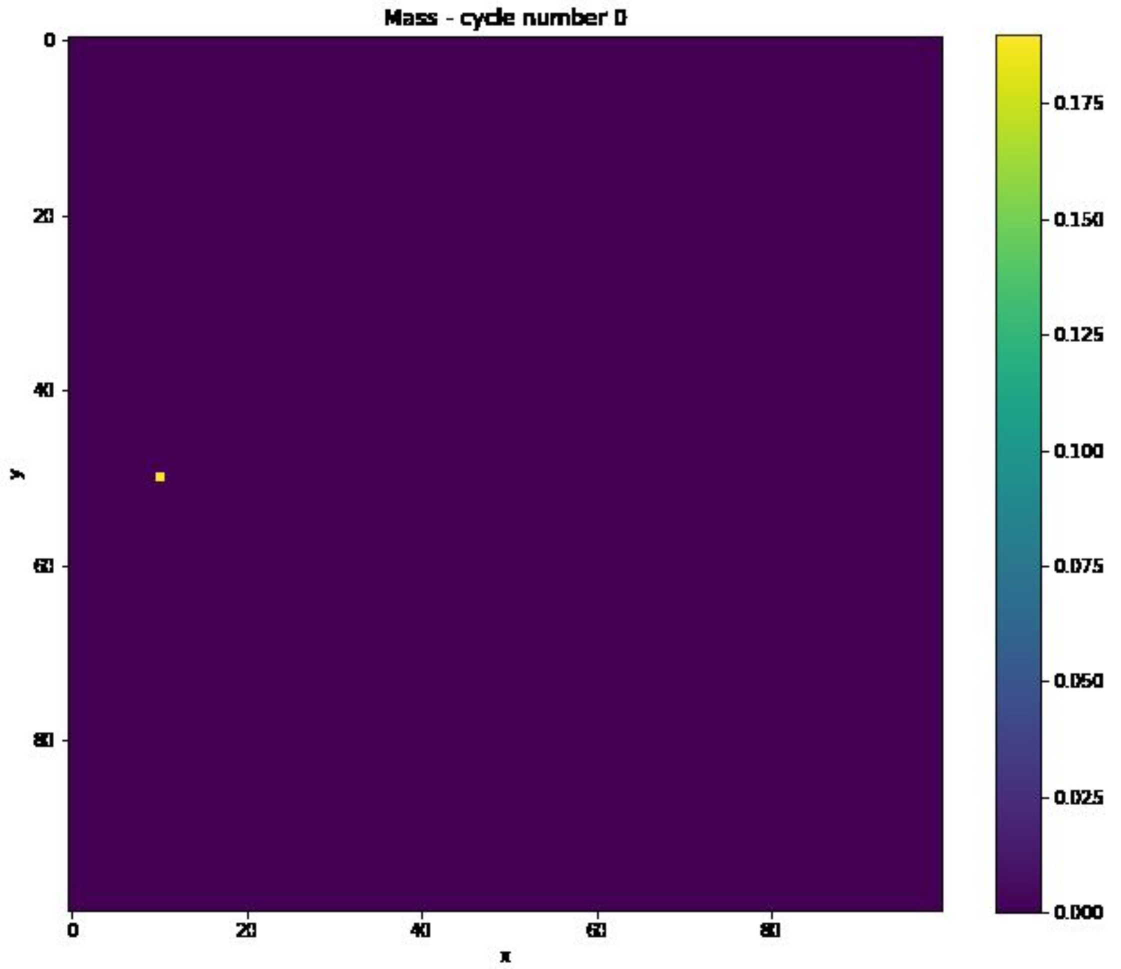
Temperature evolution with time in SGoL



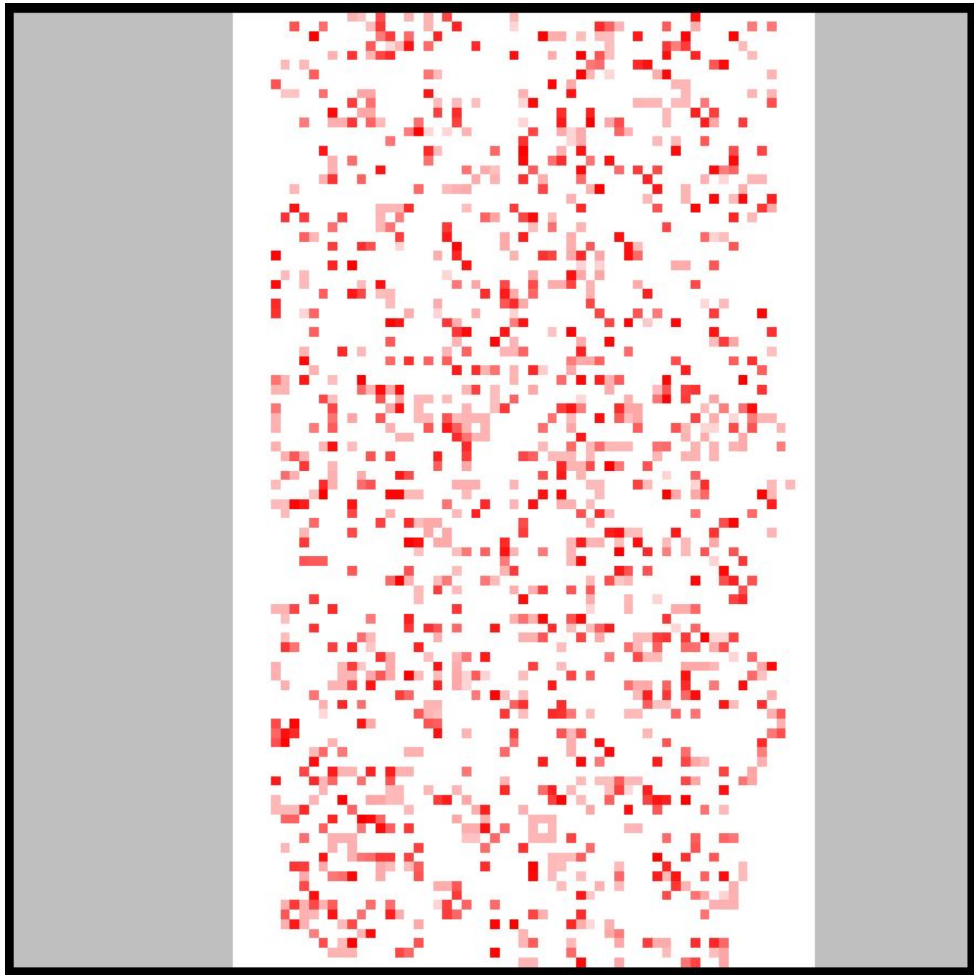


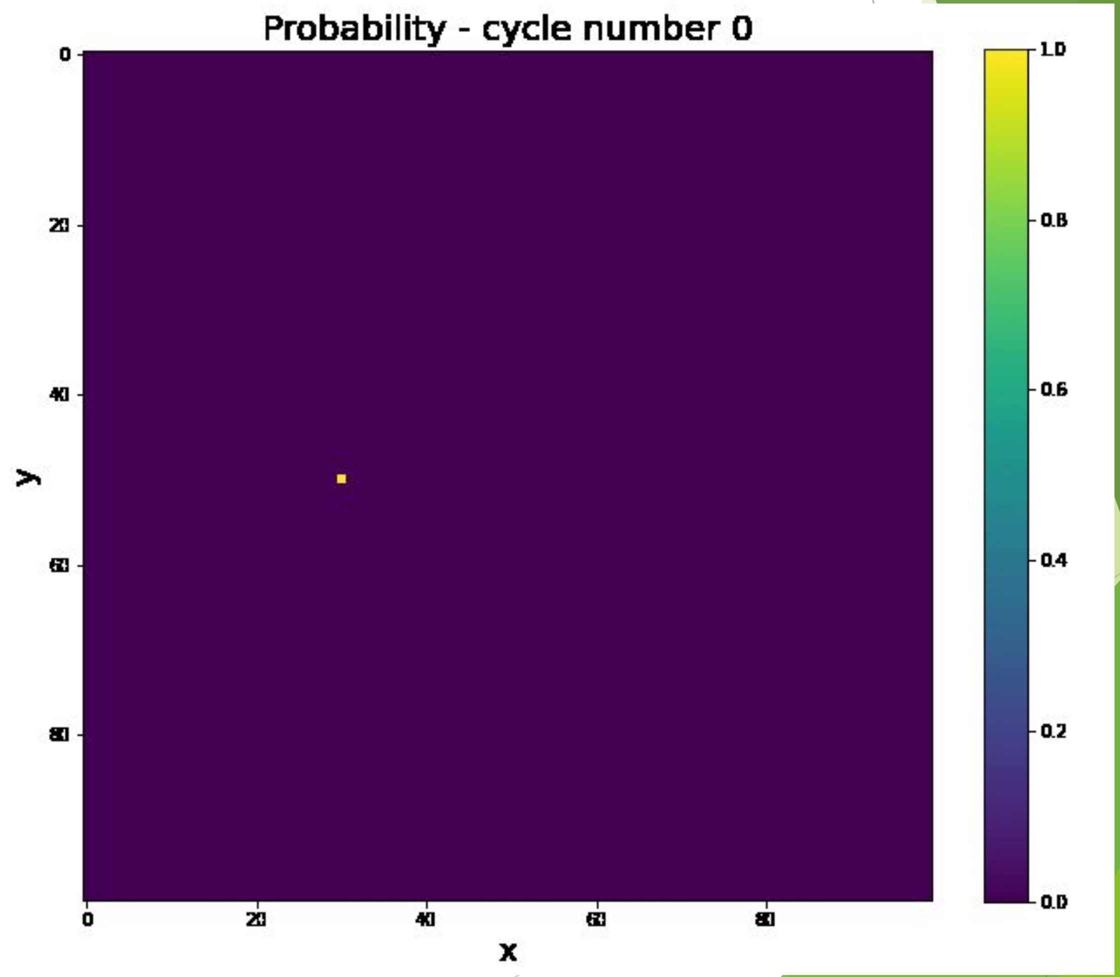
Dynamics of diffusion for a system with two barriers, with two small holes in each barrier



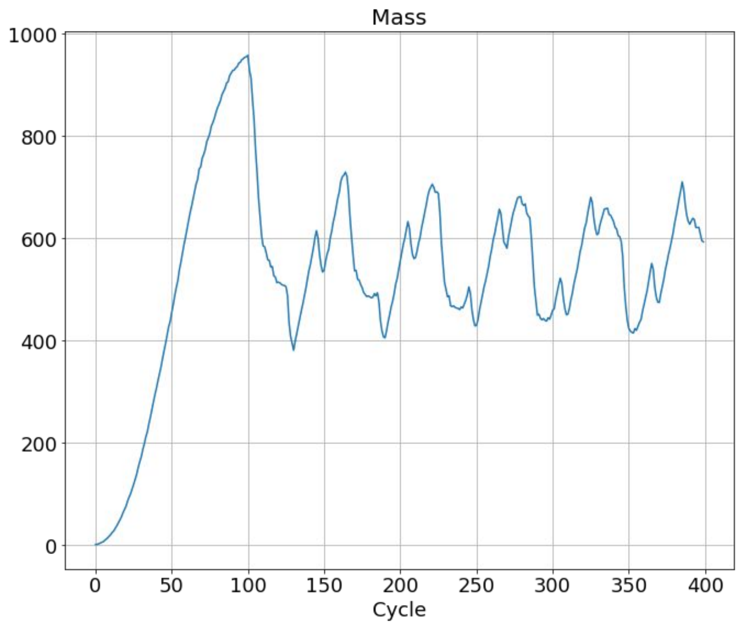


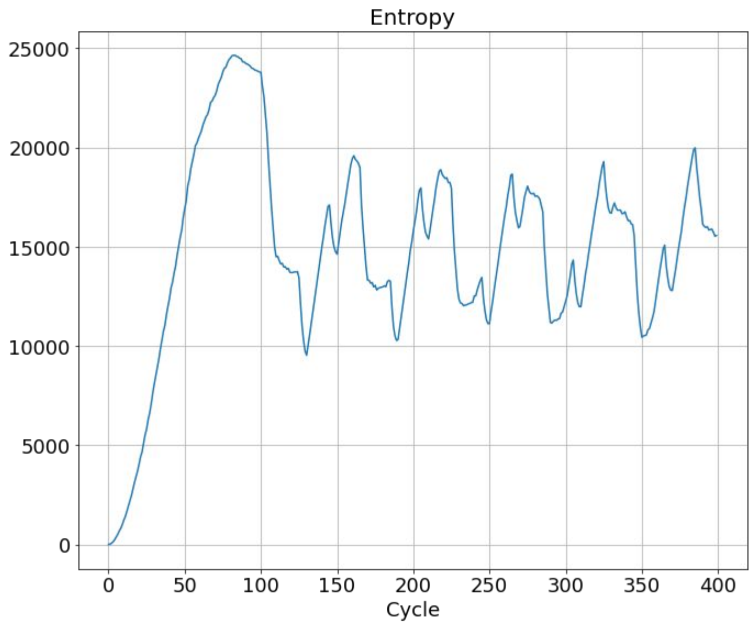
Evolution probability distribution with time in SGoL with two sinusoidally moving barriers

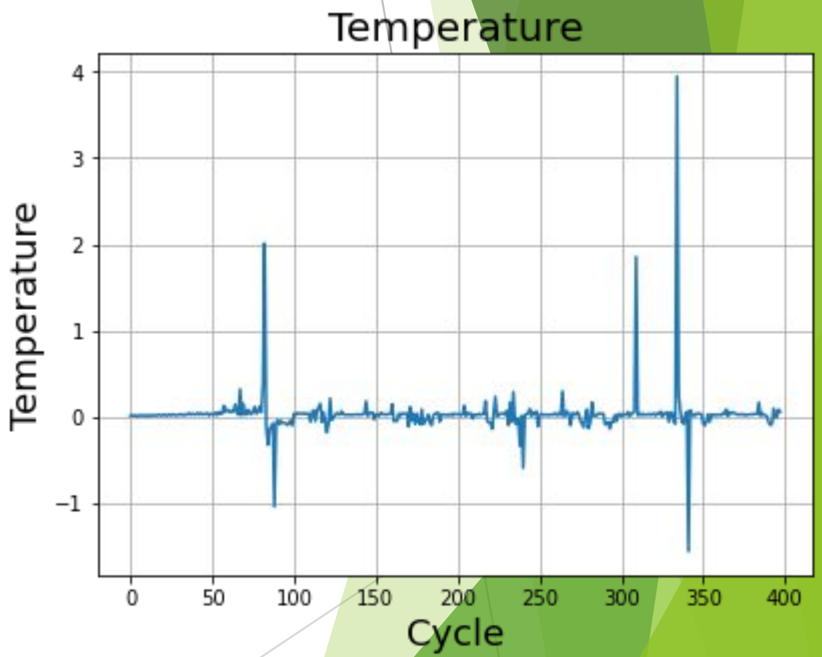




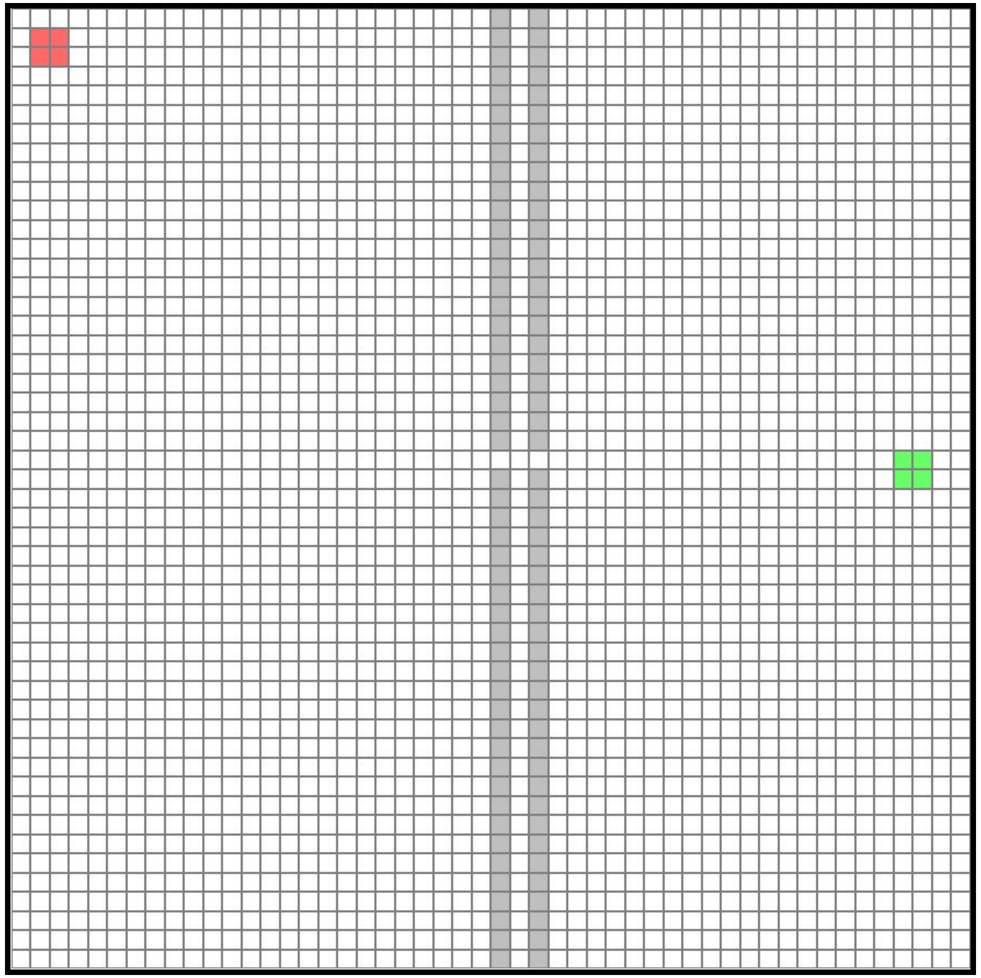
Thermodynamic parameters evolution (mass, entropy, temperature) with time in SGoL

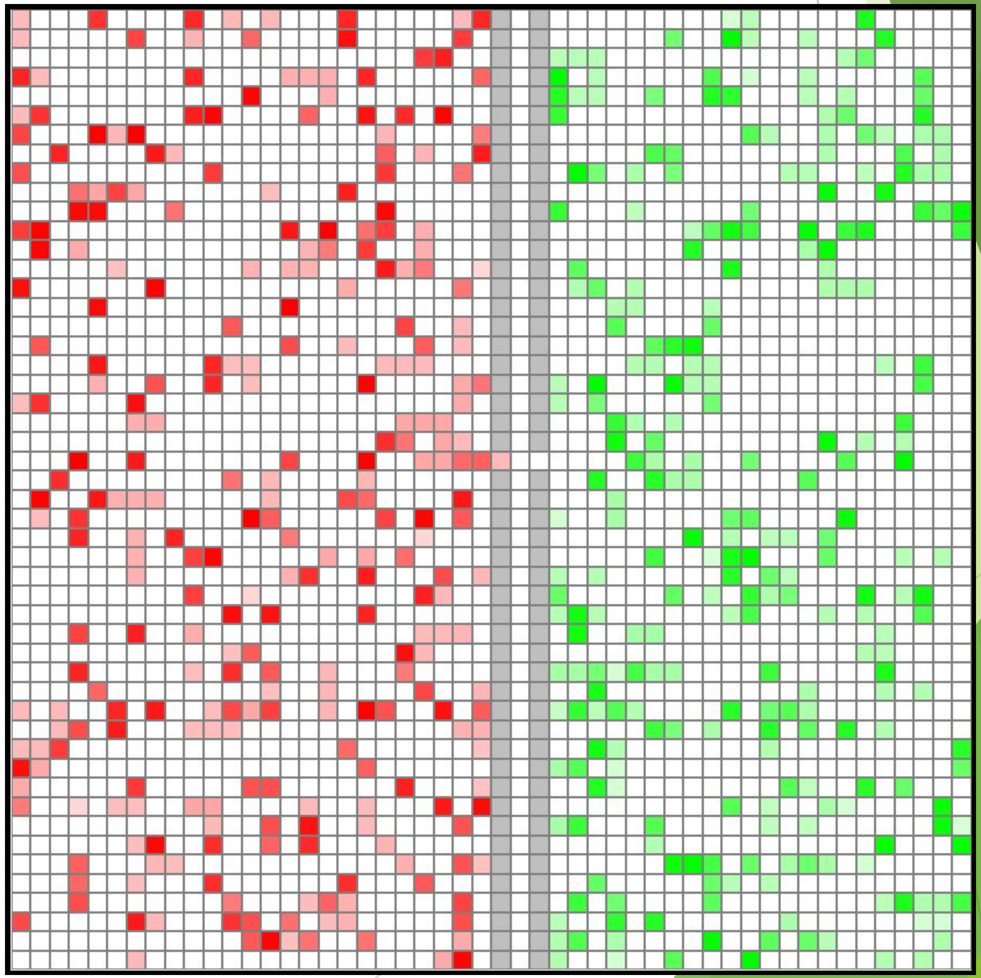






Two cellular automata tribes weakly interacting with each other via a small hole in a double barrier





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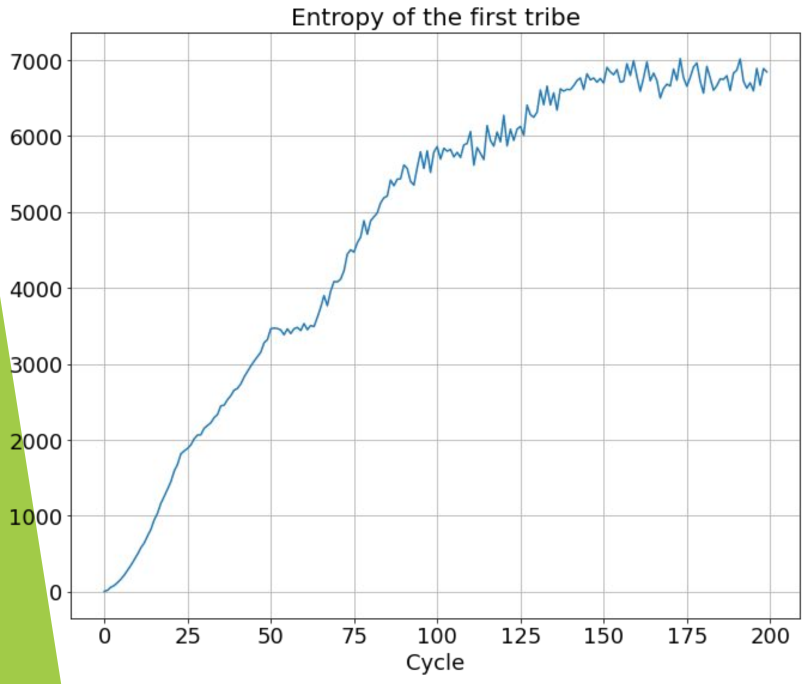
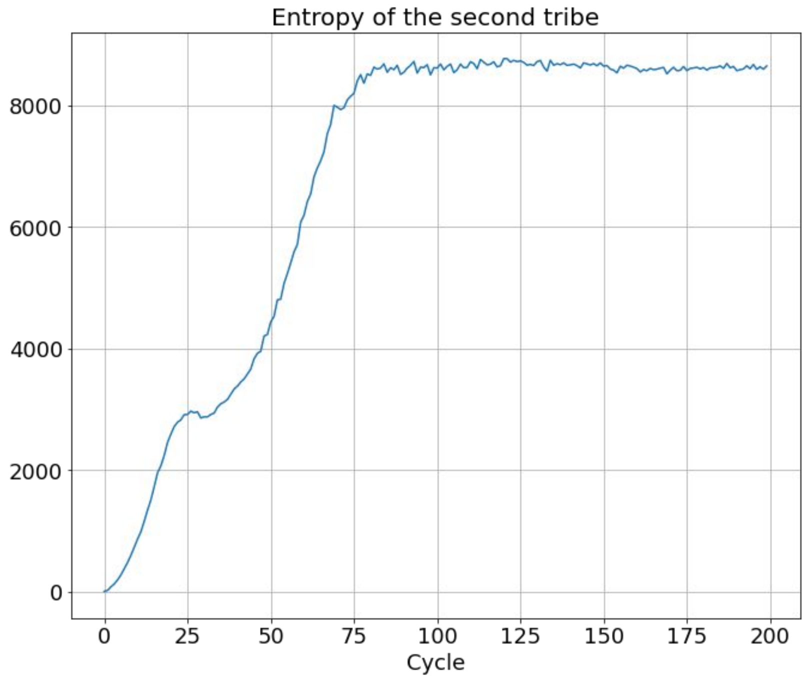
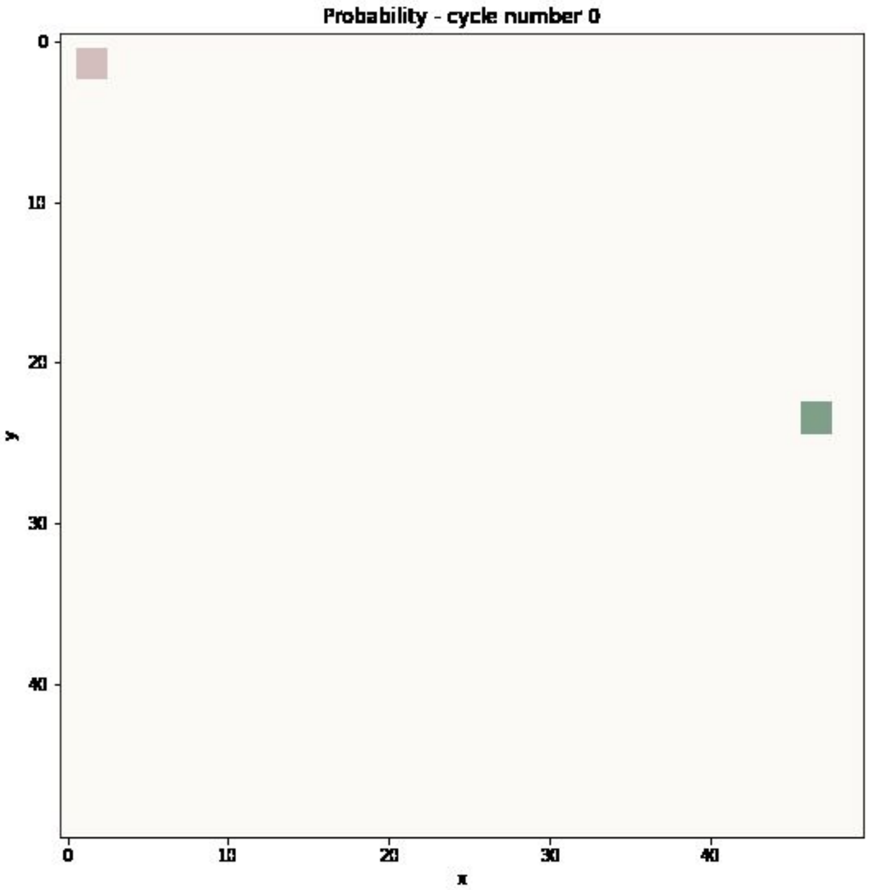






Towards the quantification of the Game of Life

- A living cell that stays alive in the next cycle has mass and phase equal to

$$m_i(t+1) = |m_i(t)| * e^{i\varphi_i(t)} = |m_i(t)| * \left(cos\left(\varphi_i(t)\right) + i\, sin\left(\varphi_i(t)\right)\right)$$

$$\varphi_i(t+1) = \varphi_i(t) + k$$

$$m_i \in \mathbb{C}$$

- A dead cell that comes alive in the next cycle has mass and phase equal to

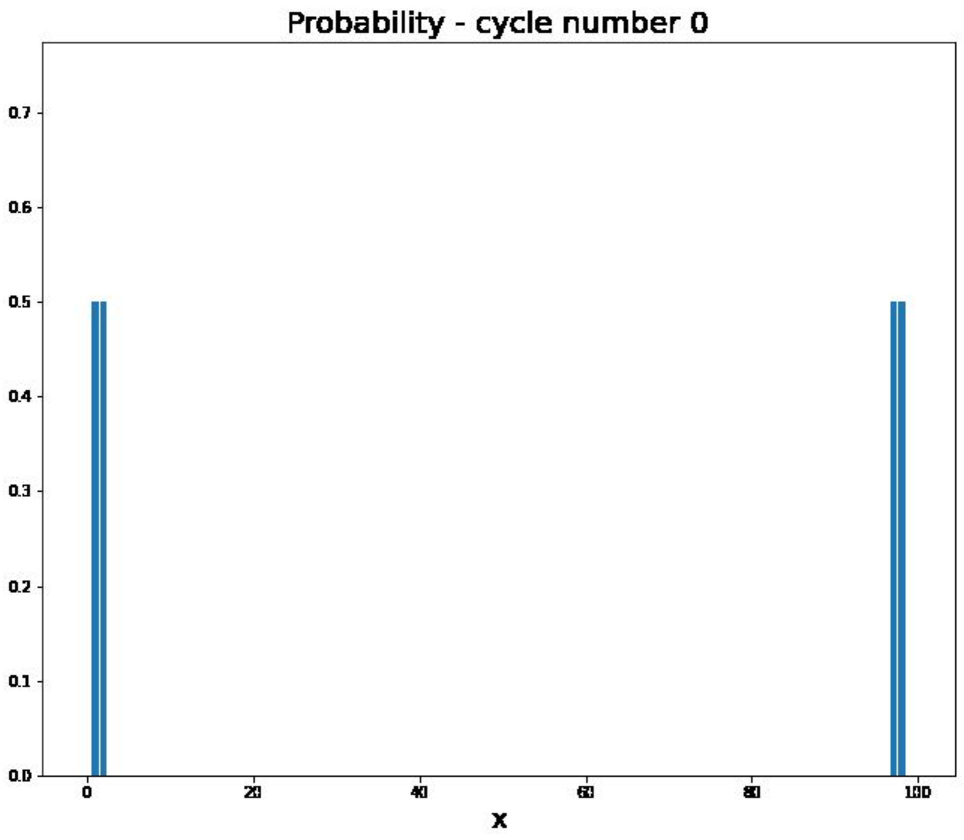
$$m_i(t+1) = \frac{1}{N}\sum_i^N M_i$$

M_i - the mass of the ith neighbour

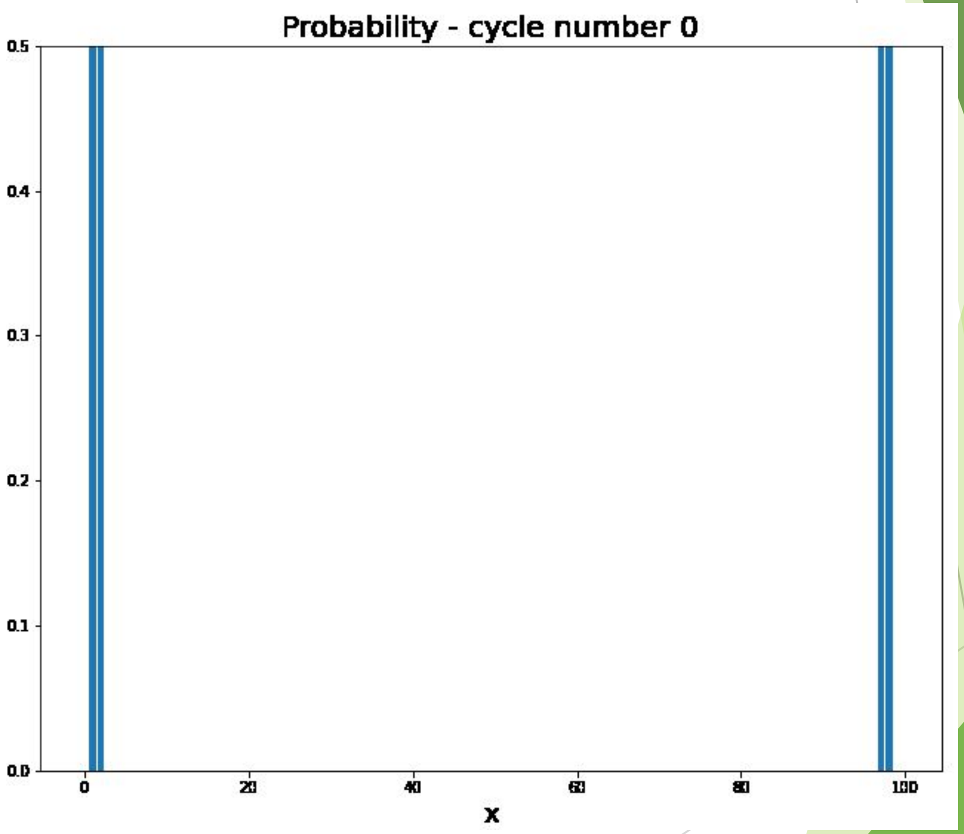
$$\varphi_i(t+1) = \frac{1}{N}\sum_i^N \phi_i$$

ϕ_i - the phase of the ith neighbour

Evolution probability distribution with time in one dimension in SGoL



Without phase



With phase

Mapping of Stochastic Conway's Game of Life to Quantum Physics

- Classical Physics (CP):
  Fick's second law

CP: $$D(x,y,t)\left[\left(\frac{d}{dx}\right)^2+\left(\frac{d}{dy}\right)^2\right]n(x,y,t)=\frac{d}{dt}n(x,y,t)$$

QS: $$-\frac{\hbar^2}{2m}\psi(x,y,t)+\alpha(x,y,t)\psi(x,y,t)+\beta(x,y,t)|\psi(x,y,t)|=i\hbar\eta(x,y,t)\frac{d}{dt}\psi(x,y,t)$$

$$\psi(x,y,t)=\sqrt{n(x,y,t)}\exp\left(i\Theta(x,y,t)\right)$$

- Quantum System (QS)
  Hamiltonian with phase addition

Mapping of Stochastic Conway's Game of Life to Quantum Physics

$$\psi(x,t) = \sqrt{p(x,t)}e^{i\Theta(x,t)}$$

$$p(x,t) = \psi^2(x,t)e^{-2i\Theta(x,t)}$$

$$D(x,t)\frac{d^2}{dx^2}p(x,t) = \frac{d}{dt}p(x,t)$$

$$D(x,t)\frac{d^2}{dx^2}\left(\psi^2(x,t)e^{-2i\Theta(x,t)}\right) = \frac{d}{dt}\left(\psi^2(x,t)e^{-2i\Theta(x,t)}\right)$$

$$D(x,t)\frac{d^2}{dx^2}\left(\psi^2(x,t)e^{-2i\Theta(x,t)}\right) = 2\dot{\psi}(x,t)\psi(x,t)e^{-2i\Theta(x,t)} - 2i\dot{\Theta}(x,t)\psi^2(x,t)e^{-2i\Theta(x,t)}$$

$$2i\dot{\Theta}(x,t)\psi(x,t) + \frac{D(x,t)}{\psi(x,t)}e^{2i\Theta(x,t)}\frac{d^2}{dx^2}\left(\psi^2(x,t)e^{-2i\Theta(x,t)}\right) = 2\frac{d}{dt}\psi(x,t)$$

$$-\hbar\dot{\Theta}(x,t)\psi(x,t) + i\hbar\frac{D(x,t)}{2\sqrt{p(x,t)}}e^{i\Theta(x,t)}\frac{d^2}{dx^2}\left(\psi^2(x,t)e^{-2i\Theta(x,t)}\right) = i\hbar\frac{d}{dt}\psi(x,t)$$

Mapping of Stochastic Conway's Game of Life to Quantum Physics

$$i\hbar\frac{d}{dt}\psi(x,t) = -\hbar\dot{\Theta}(x,t)\psi(x,t) + i\hbar\frac{D(x,t)}{2\sqrt{p(x,t)}}e^{i\Theta(x,t)}\Bigg((2\psi_{,x}^2(x,t)e^{-2i\Theta(x,t)}$$

$$+2\psi(x,t)\psi_{,x,x}(x,t)e^{-2i\Theta(x,t)} - 8i\Theta_{,x}(x,t)\psi(x,t)\psi_{,x}(x,t)e^{-2i\Theta(x,t)} - 6i\Theta_{,x,x}(x,t)\psi^2(x,t)e^{-2i\Theta(x,t)}\Bigg)$$

$$i\hbar\frac{d}{dt}\psi(x,t) = \Bigg[i\hbar\frac{D(x,t)}{2\sqrt{p(x,t)}}e^{i\Theta(x,t)}\Bigg(\frac{2\psi_{,x}^2(x,t)}{\sqrt{p(x,t)}}e^{-3i\Theta(x,t)} + 2\psi_{,x,x}(x,t)e^{-2i\Theta(x,t)}$$

$$-8i\Theta_{,x}(x,t)\psi_{,x}(x,t)e^{-2i\Theta(x,t)} - 6i\Theta_{,x,x}(x,t)\psi(x,t)e^{-2i\Theta(x,t)}\Bigg) - \hbar\dot{\Theta}(x,t)\Bigg]\psi(x,t)$$

$$\hat{H}(x,t) = i\hbar\frac{D(x,t)}{2\sqrt{p(x,t)}}e^{i\Theta(x,t)}\Bigg(\frac{2\psi_{,x}^2(x,t)}{\sqrt{p(x,t)}}e^{-3i\Theta(x,t)} + 2\psi_{,x,x}(x,t)e^{-2i\Theta(x,t)}$$

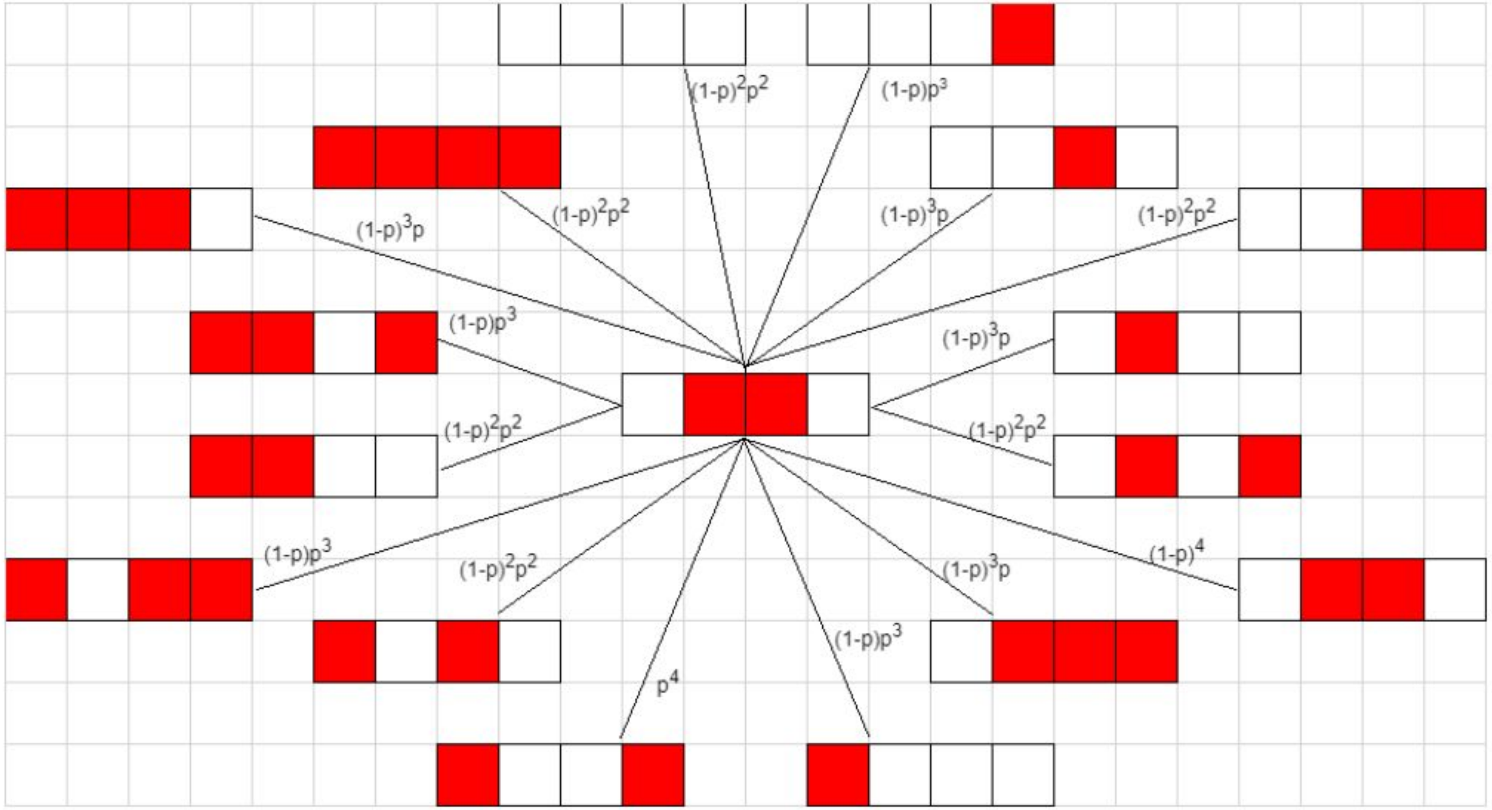
$$-8i\Theta_{,x}(x,t)\psi_{,x}(x,t)e^{-2i\Theta(x,t)} - 6i\Theta_{,x,x}(x,t)\psi(x,t)e^{-2i\Theta(x,t)}\Bigg) - \hbar\dot{\Theta}(x,t)$$

Markov Process in Stochastic Conway Game of Life

Possible scenarios of cell evolution assuming that the initial structure is a permanent structure, the cell has a probability of changing the state to the opposite equal to p, the cell has a probability of maintaining the state equal to 1-p.

The probability of occurrence of a given structure depends only on the states of the cells of the previous structure.

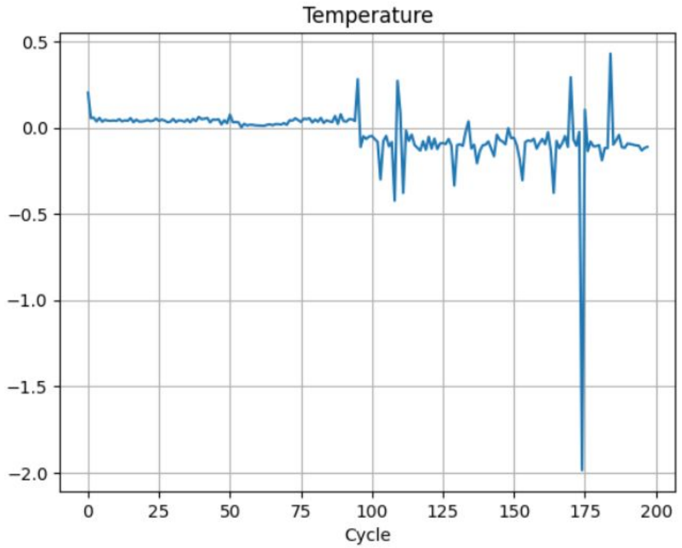
$$P(X_{m+1} = j | X_m = i, X_{m-1} = i_{m-1}, \dots, X_1 = i_1) = P(X_{m+1} = j | X_m = i)$$

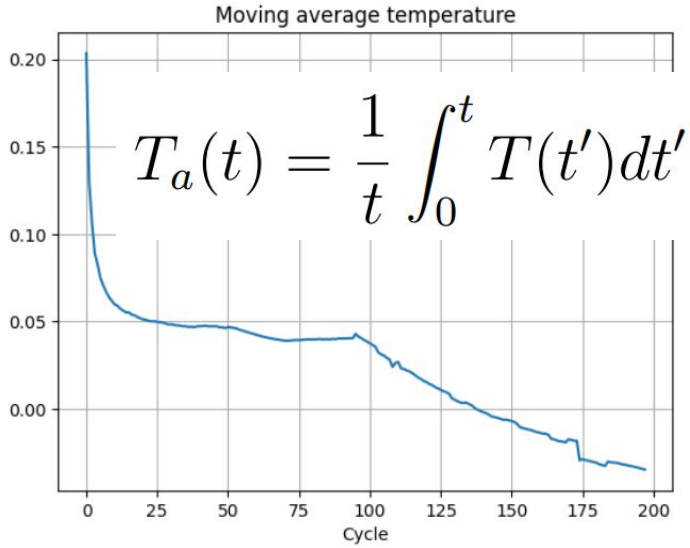


Quantum Mechanics vs Classical Statistical Physics

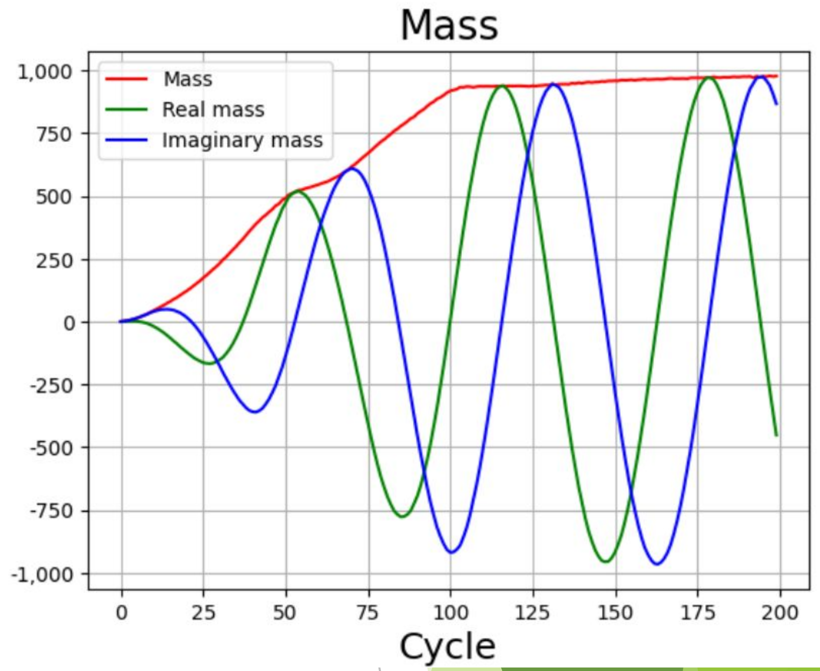
Statistical Mechanics	**Quantum Mechanics**
states: $x \in X$	histories: $x \in X$
probabilities: $p\colon X \to [0,\infty)$	amplitudes: $a\colon X \to \mathbb{C}$
energy: $E\colon X \to \mathbb{R}$	action: $A\colon X \to \mathbb{R}$
temperature: T	Planck's constant times i: $i\hbar$
coolness: $\beta = 1/T$	classicality: $\lambda = 1/i\hbar$
partition function: $Z = \int_X e^{-\beta E(x)}\, dx$	partition function: $Z = \int_X e^{-\lambda A(x)}\, dx$
Boltzmann distribution: $p(x) = e^{-\beta E(x)}/Z$	Feynman sum over histories: $a(x) = e^{-\lambda A(x)}/Z$
entropy: $S = -\int_X p(x) \ln p(x)\, dx$	quantropy: $Q = -\int_X a(x) \ln a(x)\, dx$
expected energy: $\langle E \rangle = \int_X p(x) E(x)\, dx$	expected action: $\langle A \rangle = \int_X a(x) A(x)\, dx$
free energy: $F = \langle E \rangle - TS$	free action: $\Phi = \langle A \rangle - i\hbar Q$
$\langle E \rangle = -\frac{d}{d\beta} \ln Z$	$\langle A \rangle = -\frac{d}{d\lambda} \ln Z$
$F = -\frac{1}{\beta} \ln Z$	$\Phi = -\frac{1}{\lambda} \ln Z$
$S = \ln Z - \beta \frac{d}{d\beta} \ln Z$	$Q = \ln Z - \lambda \frac{d}{d\lambda} \ln Z$
principle of maximum entropy	principle of stationary quantropy
principle of minimum energy (in $T \to 0$ limit)	principle of stationary action (in $\hbar \to 0$ limit)

Complex value Conway Game of Life

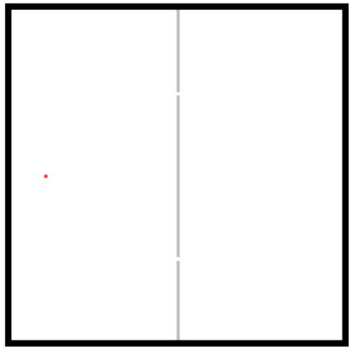


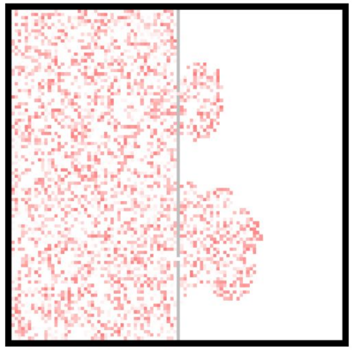


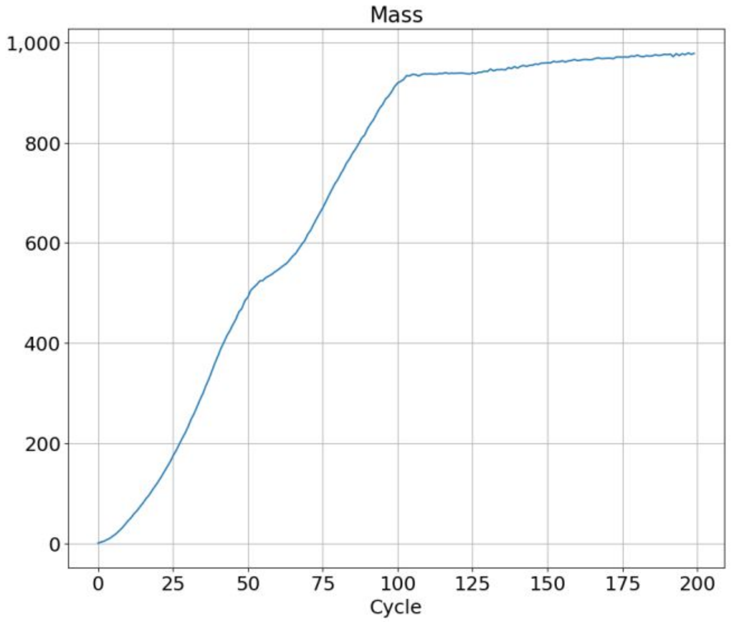
$$T_a(t) = \frac{1}{t}\int_0^t T(t')dt'$$

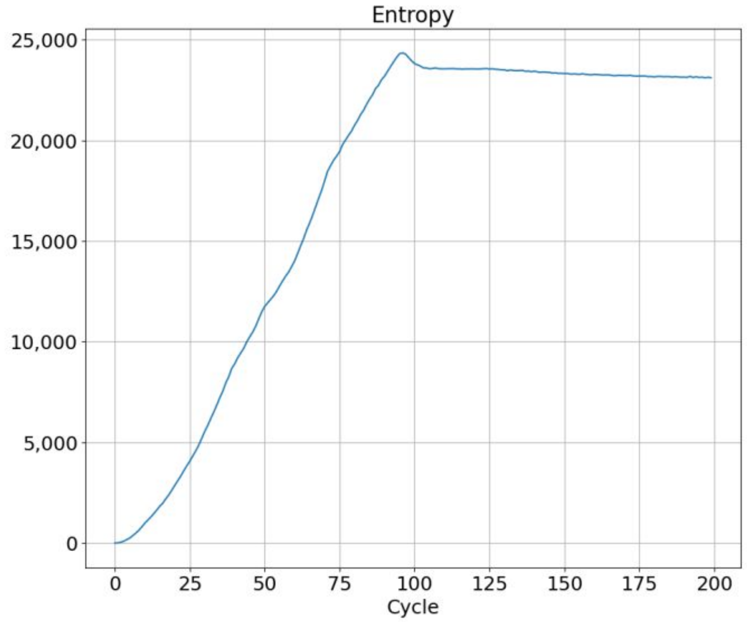


If the total weight of the living neighbors cell is less than 0.3 or greater than 1.0, the cell changes its state to dead in the next time step. If the total mass of the dead cell's neighbors is less than 0.45 or greater than 1.0, the cell does not change its state in the next time step.



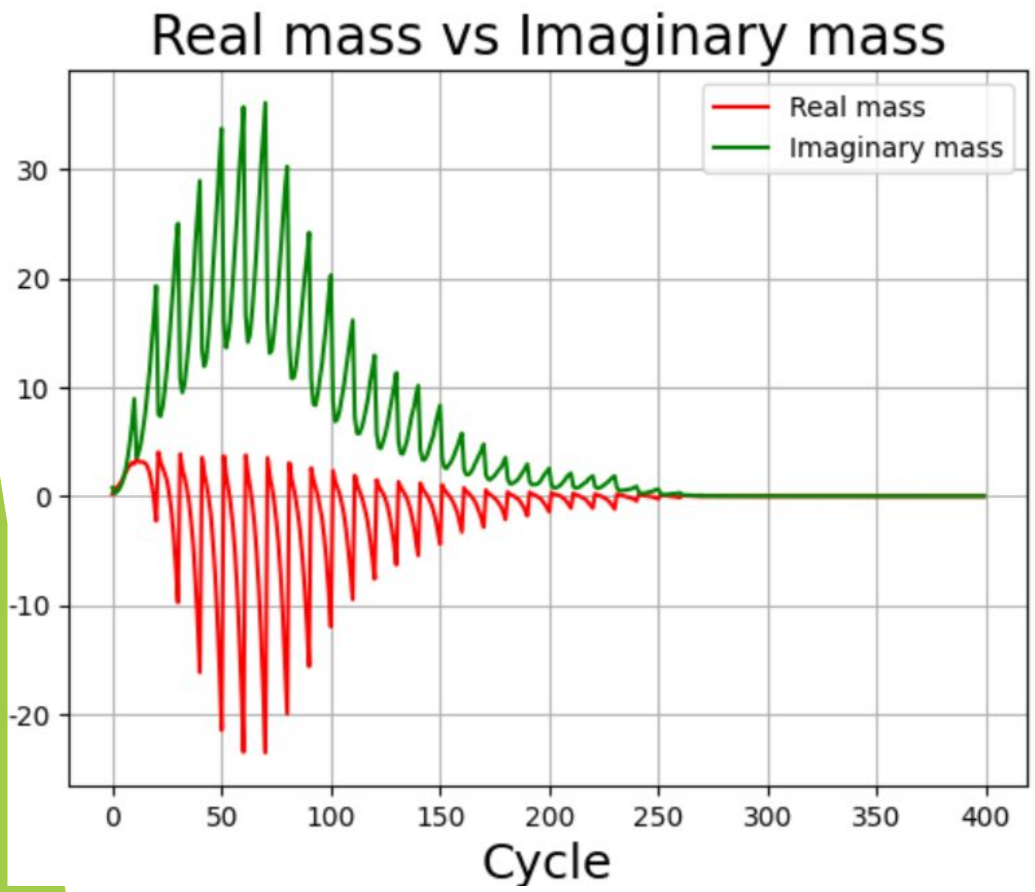


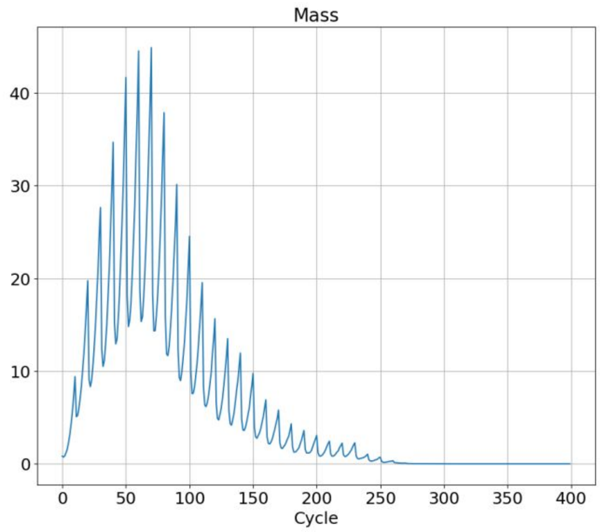


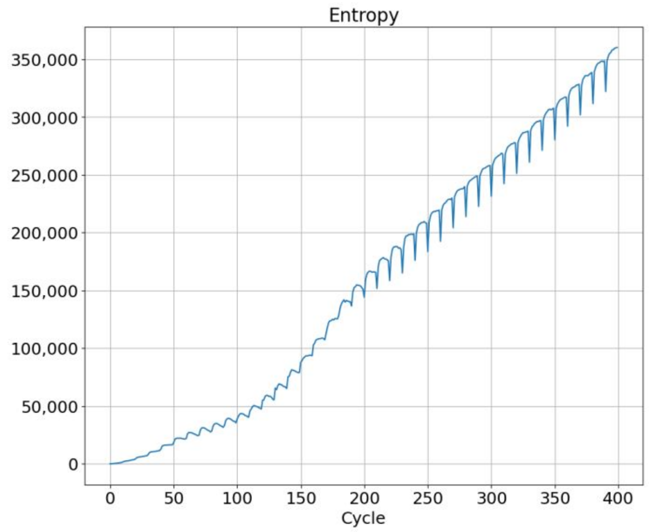


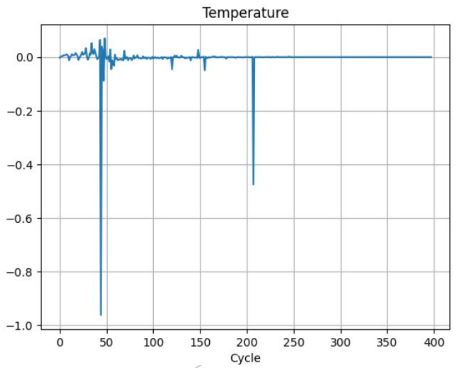
Averaged Complex value Conway Game of Life

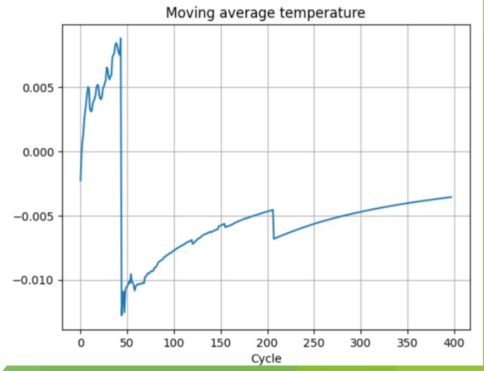
The only difference is the introduced mass averaging and cell phases every tenth step of the simulation. During this process it is counted the sum of the masses and phases of four adjacent cells in the shape of a 2x2 square. These values are then distributed equally among the cells that make up the square.











Complex value tight-binding model describing complex value Conway Game of Life

Non-hermicity of Hamiltonian

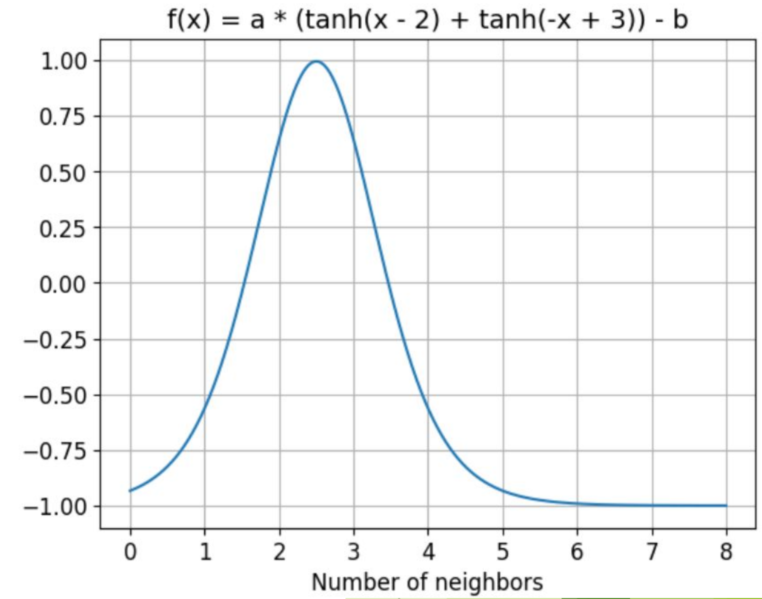
Is exploited to account for

Creationism / Annihilationism .

$$i\hbar\frac{d}{dt}\left|\Psi\right\rangle_t = \hat{H}\left|\Psi\right\rangle_t$$

:

$$\hat{H} = \sum_{k=-\infty}^{k=+\infty}\sum_{l=-\infty}^{l=+\infty}\sum_{m=-\infty}^{m=+\infty}\sum_{n=-\infty}^{n=+\infty}\left|k,l\right\rangle\left\langle m,n\right| \cdot f(k,l,m,n)$$



Quantum state killing caused by too few neighbors

$$\hat{H}(k,l) = \left|k,l\right\rangle\left\langle k,l\right|(-i)[\tanh(1-|p_1|+\cdots+|p_8|)+\tanh(|p_1|+\cdots+|p_8|)]$$

$$\hat{H}(k,l) = -\left|k,l\right\rangle\left\langle k,l\right|\lambda[\tanh(-3+|p_1|+\cdots+|p_8|)+\tanh(2-|p_1|-\cdots-|p_8|)$$

$$\hat{H}(k,l) = -\left|k,l\right\rangle\left\langle k,l\right|(-1)^{0.5}[e^{-(|p_1|+\cdots+|p_8|-3)}]$$

Quantum state killing caused by too many neighbors

Summary of obtained results in SGoL

1) Identification of thermodynamically defined temperature as proper measure of system evolution with '-' sign for systems in equilibrium
2) Identification of mass as effective energy of system (in first approximation)
3) Identification of Shannon Entropy as effective system entropy (in first approximation)
4) Generalization of Stochastic Conway's Game of Life to 4-tribe system (approximated analogy to 4-body Quantum Physics)
5) Identification of Stochastic Conway's Game of Life mapping procedure to time-dependent Schrödinger Equation
6) Formulation of hypothesis of effective evolution of SGoL expressed by non-linear second Fick law
7) Testing the concept of thermodynamic cycle applied to SGoL with moving barrier (entropy can be increased or decreased by moving wall)
8) Confirmation validity of second law of thermodynamics in SGoL (entropy maximises and saturates)
9) Identification of Shannon Entropy peak that later minimizes and saturates in SGoL

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Thank You for your attention!

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